Problem 1: Stability (25 points)

(a) What are the equilibrium states of the differential equation
\[
\ddot{x} + \sin x = 0
\] (1)
Are they (i) stable, (ii) uniformly stable, (iii) uniformly asymptotically stable, or (iv) uniformly asymptotically stable in the large?

(b) For the equation
\[
\ddot{x} + 3\dot{x} + 2x = 0
\] (2)
show that \( V = x^2 + \frac{1}{2} \dot{x}^2 \) is a Lyapunov function. Is the equilibrium state of (2) (i) uniformly stable, (ii) uniformly asymptotically stable, (iii) u.a.s.l.?

(c) Let \( A \) be a constant matrix in \( \mathbb{R}^{n \times n} \). For the equation
\[
\dot{x} = Ax
\]
what are the conditions on \( A \) under which
\[
\lim_{t \to \infty} x(t) = 0
\]
If \( A(t) \) is a time-varying matrix, with all eigenvalues in the left-half complex plane, and \( \dot{x} = A(t)x \), can you conclude that \( \lim_{t \to \infty} x(t) = 0 \)?

(d) Consider the differential equation
\[
\dot{x}(t) = (e^{\Omega t}Be^{-\Omega t})x(t) \quad x(t_0) = x_0
\] (3)
where \( \Omega \) and \( B \) are constant matrices. The solution of (3) is given by
\[
x(t) = e^{\Omega t}e^{(-\Omega+B)t}x_0
\]
Use this result to answer the following: A rotating mechanical system can be described by the differential equation
\[
\dot{x}(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} -1 & -3 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} x(t) = A(t)x(t); \quad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]
What are the eigenvalues of \( A(t) \)? What is the behavior of the solutions as \( t \to \infty \)? What can you conclude about solutions of time-varying differential equations?
Problem 2: Output-Feedback Control (30 points)

Consider the unknown plant

\[ m\ddot{y} + \beta \dot{y} + ky = u \]  

(4)

where \( k \) and \( \beta \) are unknown scalars and \( m \) is a known scalar with \( m > 0 \). The goal is to design \( u \) so that \( y \) tracks a desired trajectory \( y_d \) as \( t \to \infty \). Assume that both \( \dot{y}_d \) and \( \ddot{y}_d \) are available signals. Design an adaptive phase lead controller for the following two scenarios

(a) \( y \) and \( \dot{y} \) are measurable

(b) only \( y \) is measurable

Problem 3: CRM and Disturbances (25 points)

Consider the unknown plant

\[
\dot{x} = Ax + b\lambda u + d \\
y = c^\top x
\]

where \( c, b \in \mathbb{R}^n \) are completely known, \( A \in \mathbb{R}^{n \times n} \) is unknown, \( \lambda \) is unknown with a known sign, \( x \) is accessible for measurement, \( d \) is a constant unknown disturbance, and \( (A, b) \) is controllable. The reference model is chosen as

\[
\dot{x}_m = A_m x + b r + \ell (y - y_m) \\
y_m = c^\top x_m
\]

You can assume that there exists a \( \theta^* \) such that \( A + b\theta^* = A_m \).

(a) Assume that \( d = 0 \) and \( \ell \neq 0 \). How would you choose \( \ell \) such that a globally stable adaptive controller can be designed such that \( x \) tracks \( x_m(t) \) asymptotically?

(b) Determine an adaptive controller for the above problem (when \( d = 0 \) and \( \ell \neq 0 \)). Prove that it is stable and the tracking goal mentioned in (a) is satisfied.

(c) Suppose that \( d \neq 0 \) and \( \ell \neq 0 \). Design an adaptive controller that still ensures stability and tracking. Prove that your controller meets the above goals, i.e. \( x \to x_m \) asymptotically.

Problem 4: Adaptive Control with Multiple Inputs (25 points)

Consider the multi-input plant given by

\[
\dot{x} = Ax + B\Lambda^* u
\]

(5)

where \( A \) and \( B \) are known, \( A \) is Hurwitz, and \( \Lambda^* \) is a general unknown matrix of the form

\[
\Lambda^* = \begin{bmatrix}
\lambda_1^* & \lambda_2^* \\
\lambda_2^* & \lambda_3^*
\end{bmatrix}
\]

(6)
Dimensions of these matrices are given below:

\[ A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times m}, \quad \Lambda^* \in \mathbb{R}^{m \times m} \quad (7) \]

Suppose a reference model is given by

\[ \dot{x}_m = A_m x_m + Br \quad (8) \]

where \( A_m \) has eigenvalues faster than \( A \) and satisfies the matching condition as \( A_m = A + BK^* \) for a \( K^* \in \mathbb{R}^{m \times n} \).

Answer the following questions:

1. Make appropriate assumption(s) and design an adaptive control input \( u \) such that \( x \to x_m \) when \( r(t) \in \mathbb{R}^{m \times 1} \) is a piecewise continuous function, so that (i) the adaptive controller must have smooth transients and (ii) the closed-loop adaptive system must have bounded solutions.

2. Numerically simulate the above adaptive system. You may assume that \( r(t) = [5 \sin(t), 5 \cos(t)]^T \), and

\[
A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -1 & 2 \end{bmatrix}, \quad \Lambda^* = \begin{bmatrix} 1.5 & 0.2 \\ 0.2 & 0.5 \end{bmatrix} \quad (9)
\]

You may choose an appropriate \( A_m \). Your simulation should have initial conditions for \( x(0) = [2, 2, 2]^T \) and \( x_m(0) = [0, 0, 0]^T \). Your plots must include the states \( x, x_m \), the commands \( r \), the inputs \( u \), and the adaptive control parameters as functions of time. The plots must show that your adaptive controller has smooth transients, that \( x \to x_m \) asymptotically, and that \( |x(t) - x_m(t)| < 0.2 \) for \( t > 10 \) sec.