2.153 Adaptive Control
Lecture 2
Simple Adaptive Systems: Identification and Control

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Introduction

Last time:
- Parameter identification
  - Algebraic (error model 1) and dynamic (error model 3)
  - Scalar and vector parameter systems
  - Non-recursive and recursive
- Stability
  - Introduction to using Lyapunov functions

Today:
- Identification of multiple parameters in first order plant
  - Error model 1 and 3
  - Determining update law using Lyapunov functions
- Adaptive control
Error Models

See page 273.

- Relate parameter errors (cannot measure) to output/measurement error (which can be measured)
- The stability and performance of an adaptive system is dependent upon the evolution of these errors
- An error model is the mathematical model which describes the evolution of these errors
- By using error models, we can understand and solve adaptive control problems more easily, as the error models are independent of the specific adaptive system
Identification of a Vector Parameter in an Algebraic System

Problem:

\[ y(t) = \theta^T u(t) \]

- \( \theta \): unknown
- \( y(t), u(t) \): measured

Construct a parameter estimate \( \hat{\theta} \)
Identification of a Vector Parameter in an Algebraic System

Difference the two output signals as

\[ e = \hat{y} - y \]

Express the parameter error as

\[ \tilde{\theta} = \hat{\theta} - \theta \]

Reduces to Error Model 1:

\[ u(t) \rightarrow \tilde{\theta}^T \rightarrow e(t) \]
Identification of a Vector Parameter in an Algebraic System

Error model 1:
\[ e = \tilde{\theta}^\top u \]

Propose the candidate Lyapunov function:
\[ V(\tilde{\theta}) = \tilde{\theta}^\top \tilde{\theta} \]

Time differentiating
\[ \dot{V} = 2\tilde{\theta}^\top \dot{\tilde{\theta}} \]

Adaptive law
\[ \dot{\tilde{\theta}} = -eu \]

Gives
\[ \dot{V} = -2\tilde{\theta}^\top eu \]
\[ = -2(\tilde{\theta}^\top u)^2 \]

Thus \( \dot{V} \leq 0 \Rightarrow \text{stability} \)
Parameter Identification: Motivating Example

Transfer function of a DC motor:

\[ \frac{\omega}{V} = \frac{K}{Js + B} = \frac{a_1}{s + \theta_1} \]

\( V \): Voltage input
\( \omega \): Angular Velocity output
\( K, J, B \): unknown physical parameters

Express the plant transfer function as:

\[ a_1, \theta_1 \] unknown
Parameter Identification: Motivating Example

The differential equation describing the DC motor is

\[ \dot{\omega} = -\theta_1 \omega + a_1 V \]

- The DC motor is a first-order dynamical system
- Recall: last time we assumed \( a_1 \) was known
- We looked at two procedures for identification
  - Error model 1
  - Error model 3
- Now we will consider both \( a_1 \) and \( \theta_1 \) to be unknown
- Will go through both procedures (error model 1 and 3) again
Identification: First order plant, two unknown parameters

Plant Transfer Function:

\[
\frac{x_p}{u} = \frac{a_1}{s + \theta_1}
\]

Express the plant transfer function as

\[
\frac{x_p}{u} = \frac{1}{s + \theta_1} = \frac{1}{s + \theta_m} \cdot \frac{s + \theta_m}{s + \theta_m} = \frac{1}{s + \theta_m} \cdot \frac{s + \theta_1}{s + \theta_m} = \frac{1}{s + \theta_m} \cdot \frac{\theta_m - \theta_1}{s + \theta_m} + \frac{s + \theta_1}{s + \theta_m} = \frac{1}{s + \theta_m} \cdot \left(1 - \frac{\theta_m - \theta_1}{s + \theta_m}\right)
\]

where \(\theta_m > 0\) is a known positive parameter selected by control designer.
Define $\theta \triangleq \theta_m - \theta_1$ and simplify this transfer function as

$$\frac{\omega}{u} = \frac{1}{s + \theta_m} \cdot \frac{1}{1 + \frac{\theta}{s + \theta_m}}$$

and realize this in the following block diagram representation, where we are using two states to represent a first order system.
Identification: First order plant, two unknown parameters

Recall: when $a_1$ known

$$y(t) \equiv x_p(t) - a_1 \phi(t) = \theta \phi_2(t)$$

$\Rightarrow$ Error Model 1, Identify $\theta$, a scalar

When $a_1$ unknown:

$$y(t) \equiv x_p(t) = a_1 \phi(t) + \theta \phi_2(t) = \bar{\theta}^\top \phi(t)$$

$\Rightarrow$ Error Model 1, Identify $\bar{\theta}$, a vector
Identification: First order plant, two unknown parameters

\[ y(t) = \bar{\theta}^\top \phi(t) \]

- We have just seen how to determine an update law for this system on slide 5: error model 1 with vector parameter
- Now we will see an alternate method of identifying the two unknown parameters
Identification: First order plant, two unknown parameters - An Alternate Method

The differential equation describing the plant is given by

\[ \dot{x}_p = -\theta_1 \omega + a_1 u \]

Generate an estimate of the plant output as follows, using parameter estimates in place of the unknown parameters

\[ \dot{\hat{x}}_p = -\hat{\theta}_1 \hat{x}_p + \hat{a}_1 u \]

Define the following output error and parameter errors

\[ e = \hat{x}_p - x_p \]
\[ \tilde{\theta}_1 = \hat{\theta}_1 - \theta_1 \]
\[ \tilde{a}_1 = \hat{a}_1 - a_1 \]
Identification: First order plant, two unknown parameters - An Alternate Method

The output error dynamics are given by

\[ \dot{e} = -\hat{\theta}_1 \hat{x}_p + \hat{a}_1 u + \theta_1 x_p - a_1 u \]

Add and subtract \( \theta_1 \hat{x}_p \)

\[ \begin{align*}
\dot{e} &= -\hat{\theta}_1 \hat{x}_p + \theta_1 \hat{x}_p - \theta_1 \hat{x}_p + \hat{a}_1 u + \theta_1 x_p - a_1 u \\
&= (\theta_1 - \hat{\theta}_1) \hat{x}_p - \theta_1 (\hat{x}_p - x_p) + \tilde{a}_1 u \\
&= -\theta_1 e - \tilde{\theta}_1 \hat{x}_p + \tilde{a}_1 u \\
&= -\theta_1 e + \tilde{\theta}^\top \phi
\end{align*} \]

where

\[ \begin{align*}
\theta &= \begin{bmatrix} \theta_1 \\ a_1 \end{bmatrix} \quad \text{and} \quad \phi &= \begin{bmatrix} -\hat{x}_p \\ u \end{bmatrix}
\end{align*} \]
Identification: First order plant, two unknown parameters - An Alternate Method

Error Model 3:

\[ \dot{e} = -\theta_1 e + \tilde{\theta}^\top \phi \]

- We saw error model 3 last lecture for a system with one unknown
- We will again determine an update law using a Lyapunov function
- Choose a quadratic function \( V \) of the dominant errors in the system

\[ V(e, \tilde{\theta}) = \frac{1}{2} \left( e^2 + \tilde{\theta}^\top \tilde{\theta} \right) \]

Goal: Choose \( \dot{\tilde{\theta}} \) so that \( \dot{V} \leq 0 \)
Identification: First order plant, two unknown parameters - An Alternate Method

Take the time derivative of $V$

$$\dot{V} = e \dot{e} + \tilde{\theta} \ddot{\theta}$$

$$= -\theta_1 e^2 + \tilde{\theta}^\top e \phi + \tilde{\theta}^\top \dot{\theta}$$

Choose

$$\ddot{\theta} = -e \phi$$

$$\Rightarrow \dot{V} = -\theta_1 e^2$$

Thus $\dot{V} \leq 0 \Rightarrow$ stability.
Adaptive Control

- Everything we have seen thus far has been identification of unknown parameters
- Now we introduce adaptive control: how to control systems with unknown parameters
Problem:

Plant:

\[ \dot{x}_p = a_p x_p + k_p u \]

Find \( u \) such that \( x_p \) follows a desired command.
A Model-reference Approach

- $x_p$: Output of a first-order system - can only follow 'smooth' signals
- Ensure $x_d$ is a 'smooth' signal essentially by filtering the desired command

Pose the problem as

$$\dot{x}_m = a_m x_m + k_m r$$

Set $a_m = -5$ and $k_m = 5$, say. Choose $r$ so that $x_m \approx x_d$.
Choose $u$ so that $e(t) \to 0$ as $t \to \infty$.

- $a_p$ unknown
- $k_p$ unknown, but with known sign
Certainty Equivalence Principle

Step 1: **Algebraic Part:** Find a solution to the problem when parameters are known.

Step 2: **Analytic Part:** Replace the unknown parameters by their estimates. Ensure stable and convergent behavior.

The use of the parameter estimates in place of the true parameters is known as the *certainty equivalence principle*. 
Certainty Equivalence Principle- Step 1

Step 1: Algebraic Part: Propose the control law

\[ u(t) = \theta_c x_p + k_c r \]

and choose \(\theta_c, k_c\) so that closed-loop transfer function matches the reference model transfer function.

\[ \dot{x}_p = a_p x_p + k_p (\theta_c x_p + k_c r) \]
\[ = (a_p + k_p \theta_c) x_p + k_p k_c r \]

Now compare this to the reference model equation

\[ \dot{x}_m = a_m x_m + k_m r \]

Desired Parameters: \(\theta_c = \theta^*\) and \(k_c = k^*\) must satisfy

\[ a_p + k_p \theta^* = a_m \quad \text{and} \quad k_p k^* = k_m \]
Certainty Equivalence Principle- Step 1

Solving for the nominal or ideal parameters

$$\theta^* = \frac{a_m - a_p}{k_p} \quad \text{and} \quad k^* = \frac{k_m}{k_p}$$

This is represented with the following block diagram
Certainty Equivalence Principle - Step 2

Step 2: **Analytic Part:** Replace the unknown parameters by their estimates. Ensure stable and convergent behavior. From Step 1, we have

\[ u(t) = \theta^* x_p + k^* r, \quad \theta^* = \frac{a_m - a_p}{k_p}, \quad k^* = \frac{k_m}{k_p} \]

Replace \( \theta^* \) and \( k^* \) by their estimates \( \theta(t) \) and \( k(t) \).

\[ u(t) = \theta(t) x_p + k(t) r \]

\[ \dot{\theta}(t) = ?? \quad \dot{k}(t) = ?? \]
Certainty Equivalence Principle - Step 2

Adaptive control input:

\[ u(t) = \theta(t)x_p + k(t)r \]

Define the parameter errors as

\[
\begin{align*}
\tilde{\theta}(t) &= \theta(t) - \theta^* \\
\tilde{k}(t) &= k(t) - k^*
\end{align*}
\]

Plug the control law into the plant equation:

\[
\begin{align*}
\dot{x}_p &= a_px_p + k_p u(t) \\
&= a_px_p + k_p \left[ \theta(t)x_p + k(t)r \right] \\
&= a_px_p + k_p \left[ \tilde{\theta}(t)x_p + \theta^*x_p + \tilde{k}(t)r + k^*r \right] \\
&= \left[ a_p + k_p\theta^* \right] x_p + k_p\tilde{\theta}(t)x_p + k_pk^*r + k_p\tilde{k}(t)r \\
&= a_mx_p + k_p\tilde{\theta}(t)x_p + k_mr + k_p\tilde{k}(t)r
\end{align*}
\]
Certainty Equivalence Principle - Step 2

Reference Model:
\[
\dot{x}_m = a_m x_m + k_m r
\]

Define the tracking error as
\[
e = x_p - x_m
\]

Error model:
\[
\begin{align*}
\dot{e} &= a_m e + k_p \tilde{\theta}(t) x_p + k_p \tilde{k}(t) r \\
&= a_m e + k_p \tilde{\theta}(t) \omega
\end{align*}
\]

\[
\omega = \begin{bmatrix} x_p \\ r \end{bmatrix}
\]

\[
\tilde{\theta} = \begin{bmatrix} \tilde{\theta}(t) \\ \tilde{k}(t) \end{bmatrix}
\]
Certainty Equivalence Principle - Step 2

\[ \dot{e} = a_m e + k_p \tilde{\theta}(t) \omega \]

This is again error model 3!