2.153 Adaptive Control
Lecture 15
Adaptive Control of Plants with $n^* = 1$

Anuradha Annaswamy

aanna@mit.edu
Adaptive Control using Output Feedback

Goal: Choose $u$ so that $e_1(t)$ tends to zero asymptotically.

$k_p$ and coefficients of $R_p(s)$ and $Z_p(s)$ are unknown.

$R_p(s)$ is monic and of degree $n$. $Z_p(s)$ is monic and of degree $m \leq n - 1$.

$R_m(s)$ is monic, Hurwitz, and of degree $n$. $Z_m(s)$ is monic and of degree $m \leq n - 1$.

Assumptions:

- **sign** $k_p$ known
- $Z_p(s)$ has roots in $C^-$. 
- Relative degree $n^* = 1 \implies m = n - 1$. 

(aanna@mit.edu)
\( n^* = 1 \): Four Cases

1. [Case (i)] \( k_p \) unknown, \( Z_p(s) = Z_m(s) \), \( R_p(s) = R_m(s) \)
2. [Case (ii)] \( k_p \) known, \( Z_p(s) \) unknown, \( R_p(s) = R_m(s) \)
3. [Case (iii)] \( k_p \) known, \( Z_p(s) = Z_m(s) \), \( R_p(s) \) unknown
4. [Case (iv)] \( k_p \) unknown, \( Z_p(s) \) and \( R_p(s) \) unknown
Case (i) $k_p$ unknown, $Z_p(s) = Z_m(s)$, $R_p(s) = R_m(s)$, $W(s) = \frac{Z_m(s)}{R_m(s)}$

Algebraic part: $k_c = k^* = \frac{k_m}{k_p}$, $u(t) = k^* r(t)$

Analytic part: $u(t) = k(t) r(t) = \left( k^* + \tilde{k}(t) \right) r(t)$

$e_1(t) = k_p W(s) u(t) - k_m W(s) r(t) = k_p W(s) \left( k^* + \tilde{k}(t) \right) r(t) - k_m W(s) r(t)$

$e_1(t) = k_p W(s) \tilde{k}(t) r(t)$
Case (i) $k_p$ unknown, $Z_p(s) = Z_m(s)$, $R_p(s) = R_m(s)$,

$W(s) = \frac{Z_m(s)}{R_m(s)}$

**Diagram:**

- **Algebraic part:** $k_c = k^* = \frac{k_m}{k_p}$, $u(t) = k^*r(t)$

- **Analytic part:** $u(t) = k(t)r(t) = \left(k^* + \tilde{k}(t)\right)r(t)$

$e_1(t) = k_pW(s)u(t) - k_mW(s)r(t) = k_pW(s)\left(k^* + \tilde{k}(t)\right)r(t) - k_mW(s)r(t)$

$e_1(t) = k_pW(s)\tilde{k}(t)r(t)$
Case (i) $k_p$ unknown, $Z_p(s) = Z_m(s)$, $R_p(s) = R_m(s)$,

$$W(s) = \frac{Z_m(s)}{R_m(s)}$$

**Algebraic part:** $k_c = k^* = \frac{k_m}{k_p}$, \[ u(t) = k^* r(t) \]

**Analytic part:** \[ u(t) = k(t) r(t) = \left(k^* + \hat{k}(t)\right) r(t) \]

\[
e_1(t) = k_p W(s) u(t) - k_m W(s) r(t) = k_p W(s) \left(k^* + \hat{k}(t)\right) r(t) - k_m W(s) r(t)
\]

\[
e_1(t) = k_p W(s) \hat{k}(t) r(t)
\]
Case (i) $k_p$ unknown, $Z_p(s) = Z_m(s)$, $R_p(s) = R_m(s)$,

$$W(s) = \frac{Z_m(s)}{R_m(s)}$$

Algebraic part: $k_c = k^* = \frac{k_m}{k_p}$, $u(t) = k^* r(t)$

Analytic part: $u(t) = k(t) r(t) = \left( k^* + \tilde{k}(t) \right) r(t)$

$$e_1(t) = k_p W(s) u(t) - k_m W(s) r(t) = k_p W(s) \left( k^* + \tilde{k}(t) \right) r(t) - k_m W(s) r(t)$$

$$e_1(t) = k_p W(s) \tilde{k}(t) r(t)$$
Case (i) Only $k_p$ unknown:

Error Model:

![System Diagram]

- Error Model 3

\[ \dot{k} = -\text{sign}(k_p)e_1r \]

$W(s)$ must be strictly positive real. \[ \Rightarrow \]

- $Z_m(s)$ must have roots in $C^-$.  
- Poles and Zeros of $W(s)$ must be interlaced.
Case (ii) $k_p$ known, $Z_p(s)$ unknown, $R_p(s) = R_m(s)$ known

$$u(t) = W_c(s)r(t) = \frac{Z_m(s)}{Z_p(s)}r(t)$$

Need the roots of $Z_p(s)$ to be in $C^-$. 
Case (ii) \( k_p \) known, \( Z_p(s) \) unknown, \( R_p(s) = R_m(s) \) known

\[ u(t) = W_c(s) r(t) = \frac{Z_m(s)}{Z_p(s)} r(t) \]

Need the roots of \( Z_p(s) \) to be in \( C^- \).
Case (ii) $k_p$ known, $Z_p(s)$ unknown, $R_p(s) = R_m(s)$ known

\[ u(t) = W_c(s)r(t) = \frac{Z_m(s)}{Z_p(s)}r(t) \]

Need the roots of $Z_p(s)$ to be in $C^-$. 
Case (ii): Realization of $W_c(s)$

\[ Z_m(s) - Z_p(s) = t_1^*(s) \]

\[ W_c(s) = \frac{Z_m(s)}{Z_p(s)} = \frac{Z_m(s)}{Z_m(s) - t_1^*(s)} \]

\[ = \frac{1}{1 - \frac{t_1^*(s)}{Z_m(s)}} \]
Case (ii): Realization of \( \frac{t_1(s)}{Z_m(s)} \)

Algebraic part: \( \frac{t_1(s)}{Z_m(s)} = \theta_{1c}^T (sI - F_{n-1})^{-1} g_{n-1} \)

\( t_1(s) = t_1^*(s) \implies \theta_{1c} = \theta_{1c}^* \)
Case (ii): Realization of \( \frac{t_1(s)}{Z_m(s)} \)

\[
\begin{align*}
\theta_1^* & \quad \theta_1^* \left( sI - F_{n-1} \right)^{-1} g_{n-1} \\
F_{n-1}, g_{n-1} & \quad u
\end{align*}
\]

Algebraic part:
\[
\frac{t_1(s)}{Z_m(s)} = \theta_1^T \left( sI - F_{n-1} \right)^{-1} g_{n-1}
\]

\[
t_1(s) = t_1^*(s) \quad \implies \theta_1 = \theta_1^*
\]
Case (ii): Analytic part

\[ \theta_1(t) = \theta_1^* + \tilde{\theta}_1(t) \]
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\[ \theta_1(t) = \theta_1^* + \tilde{\theta}_1(t) \]
Case (ii): Analytic part

\[ \theta_1(t) = \theta_1^* + \tilde{\theta}_1(t) \]
Case (ii): Representation of the Reference Model

Reference Model

\[ r \rightarrow k_m \frac{Z_m(s)}{R_m(s)} \rightarrow y_m \]
Case (ii): Representation of the Reference Model

\[ Z_m(s) \quad Z_p(s) \quad k_m \frac{Z_p(s)}{R_m(s)} \quad y_m \]

\[ r \rightarrow \quad \frac{Z_m(s)}{Z_p(s)} \quad \rightarrow \quad \frac{k_m}{R_m(s)} \quad \rightarrow \quad y_m \]
Case (ii): Representation of the Reference Model

\[
\begin{align*}
    &[r] + \\
    &\theta^*_1 \quad \omega_1 \quad F_{n-1}, \ g_{n-1} \\
    &\downarrow \quad \downarrow \quad \downarrow \\
    &u \quad k_m \frac{Z_p(s)}{R_m(s)} \quad y_m
    
\end{align*}
\]
Case (ii): Closed-loop Plant and Reference Model

$$r \rightarrow k_m \frac{Z_m(s)}{R_m(s)} \rightarrow y_m \rightarrow y_p \rightarrow \omega_1 \rightarrow \theta_1^T(t)$$

Reference Model

Plant

$$F_{n-1}, g_{n-1}$$
Case (ii): Closed-loop Plant and Reference Model

\[ r \xrightarrow{\tilde{\theta}_1^T(t)} k_m \frac{Z_m(s)}{R_m(s)} \xrightarrow{y_m} -e_1 \]

Reference Model

\[ \omega_1 \xrightarrow{\tilde{\theta}_1^T(t)} + \]

Plant

\[ F_{n-1}, g_{n-1} \]

\[ \theta_1^* \]

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Case (ii) Error Model

\[ \dot{\theta}_1 = -e_1 \omega_1 \]

- Error Model 3

- \( Z_m(s) \) must have roots in \( C^- \).
- Poles and Zeros must be interlaced.
- \( e, \tilde{\theta}_1 \) bounded. \( \lim_{t \to \infty} e_1(t) = 0 \).
Case (iii) \( k_p \) known, \( Z_p(s) = Z_m(s) \), \( R_p(s) \) unknown - Algebraic Part

Transfer function from \( r \) to \( y_p \):

\[
\frac{k_mZ_m(s)}{R_p(s)} \left( 1 - \frac{k_mZ_m(s)}{R_p(s)} \frac{t_2(s)}{Z_m(s)} \right) = \frac{k_mZ_m(s)}{R_p(s) - k_m t_2(s)}
\]

\[
k_m t_2^*(s) = R_p(s) - R_m(s)
\]

\[
t_2^*(s) = \frac{\theta_2^* + \theta_2^T(sI - F_{n-1})^{-1}g_{n-1}}{Z_m(s)}
\]
Case (iii) $k_p$ known, $Z_p(s) = Z_m(s)$, $R_p(s)$ unknown - Algebraic Part

Transfer function from $r$ to $y_p$:

$$\frac{k_m Z_m(s)}{R_p(s)} \frac{Z_m(s)}{1 - \frac{k_m Z_m(s)}{R_p(s)} t_2(s)} = \frac{k_m Z_m(s)}{R_p(s) - k_m t_2(s)}$$

$$k_m t_2^*(s) = R_p(s) - R_m(s)$$

$$\frac{t_2^*(s)}{Z_m(s)} = \theta_{20}^* + \theta_{2T}^* (s I - F_{n-1})^{-1} g_{n-1}$$
Case (iii) $k_p$ known, $Z_p(s) = Z_m(s)$, $R_p(s)$ unknown - Algebraic Part

Transfer function from $r$ to $y_p$ :

$$
\frac{\frac{k_m Z_m(s)}{R_p(s)}}{1 - \frac{k_m Z_m(s)}{R_p(s)} \frac{t_2(s)}{Z_m(s)}} = \frac{k_m Z_m(s)}{R_p(s) - k_m t_2(s)}
$$

$$
k_m t_2^*(s) = R_p(s) - R_m(s)
$$

$$
\frac{t_2^*(s)}{Z_m(s)} = \theta_{20}^* + \theta_{2T}^* (sI - F_{n-1})^{-1} g_{n-1}
$$

$$
W_c(s) = \theta_{20c} + \theta_{2c}^T (sI - F_{n-1})^{-1} g_{n-1}
$$

$$
= \frac{t_2(s)}{Z_m(s)}
$$
Case (iii) $k_p$ known, $Z_p(s) = Z_m(s)$, $R_p(s)$ unknown - Algebraic Part

$$W_c(s) = \theta_{20c} + \theta^T_{2c}(sI - F_{n-1})^{-1}g_{n-1} = \frac{t_2(s)}{Z_m(s)}$$

Transfer function from $r$ to $y_p$:

$$\frac{k_mZ_m(s)}{R_p(s)} = \frac{k_mZ_m(s)}{R_p(s) - k_m t_2(s)}$$

$$k_m t_2^*(s) = R_p(s) - R_m(s)$$

$$\frac{t_2^*(s)}{Z_m(s)} = \theta_{20c}^* + \theta_{2}^T(sI - F_{n-1})^{-1}g_{n-1}$$
Case (iii) $k_p$ known, $Z_p(s) = Z_m(s)$, $R_p(s)$ unknown - Algebraic Part

Transfer function from $r$ to $y_p$:

$$W_c(s) = \theta_{20c} + \theta_{2c}^T(sI - F_{n-1})^{-1}g_{n-1}$$

$$= \frac{t_2(s)}{Z_m(s)}$$

$$\frac{k_m Z_m(s)}{R_p(s)} \frac{1}{1 - \frac{k_m Z_m(s)}{R_p(s)}} \frac{t_2(s)}{Z_m(s)} = \frac{k_m Z_m(s)}{R_p(s) - k_m t_2(s)}$$

$$k_m t_2^*(s) = R_p(s) - R_m(s)$$

$$\frac{t_2^*(s)}{Z_m(s)} = \frac{\theta_{20}^* + \theta_{2c}^T(sI - F_{n-1})^{-1}g_{n-1}}{	heta_{20}^* + \theta_{2c}^T(sI - F_{n-1})^{-1}g_{n-1}}$$
Analytic Part

Reference Model

\[ y_m = \frac{k_m Z_m(s)}{R_m(s)} \]

Plant

\[ y_p = \frac{k_m Z_m(s)}{R_p(s)} \]

\[ e_1 = r + \omega_2 + \theta_2^\top F_{n-1}, g_{n-1} \]

\[ \theta_2^* = \begin{bmatrix} \theta_2^{*1} \\ \theta_2^{*2} \end{bmatrix} \]

\[ \theta_{20} \]

\[ \omega_2 \]

\( r \)

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Analytic Part

\[ r + \theta_2^T(t) \]

Reference Model

\[ k_m \frac{Z_m(s)}{R_m(s)} \]

Plant

\[ k_m \frac{Z_m(s)}{R_p(s)} \]

\[ F_{n-1}, g_{n-1} \]

\[ \omega_2 \]

\[ \theta_{20}(t) \]

\[ e_1 \]

\[ y_m \]

\[ y_p \]
Analytic Part

\[ \theta_2(t) \]

\[ \bar{\omega}_2 \]

\[ \bar{\theta}_2(t) \]

\[ r \]

\[ + \]

\[ k_m \frac{Z_m(s)}{R_m(s)} \]

\[ k_m \frac{Z_m(s)}{R_p(s)} \]

\[ y_m \]

\[ y_p \]

\[ e_1 \]

Reference Model

Plant

\[ F_{n-1}, g_{n-1} \]

\( (\text{aanna@mit.edu}) \)
Case (iii) Error Model

\[ \bar{\omega}_2 \xrightarrow{\tilde{\theta}_2^T} k_m \frac{Z_m(s)}{R_m(s)} \xrightarrow{e_1} \]

needs to be SPR

- Error Model 3

\[ \dot{\tilde{\theta}}_2 = -e_1 \omega_2 \]

\[ \dot{\tilde{\theta}}_{20} = -e_1 y_p \]

- \( Z_m(s) \) must have roots in \( C^- \).
- Poles and Zeros must be interlaced.
- \( e, \tilde{\theta}_2 \) bounded. \( \lim_{t \to \infty} e_1(t) = 0 \).
Case (iv) $k_p$ unknown, $Z_p(s)$, $R_p(s)$ unknown - Algebraic Part

\[ \frac{k_p Z_p}{R_p} \]

\[
\begin{align*}
Z_m(s) - t_1^*(s) &= Z_p(s), \\
R_p(s) - k_p t_2^*(s) &= R_m(s)
\end{align*}
\]

\[ k^* = \frac{k_m}{k_p} \]
Case (iv) $k_p$ unknown, $Z_p(s)$, $R_p(s)$ unknown - Algebraic Part

TF from $r$ to $y_p$:

$$k_c \cdot \frac{1}{1 - \frac{t_1}{Z_m}} \cdot \frac{Z_m \cdot k_p Z_p}{R_p}$$

$$= k_c \cdot \frac{Z_m \cdot k_p Z_p}{1 - \frac{Z_m \cdot k_p Z_p}{Z_p R_p} \cdot \frac{t_2}{Z_m R_p}} = \frac{k_c k_p Z_m}{R_p} \frac{1}{1 - k_p t_2^*/R_p}$$

$$t_1^* = \frac{k_c k_p Z_m}{R_p}$$

$$t_2^* = \frac{k_c k_p Z_m}{R_m} = \frac{k_m Z_m}{R_m}$$
Case (iv) $k_p$ unknown, $Z_p(s)$, $R_p(s)$ unknown - Algebraic Part (contd).

\[
\frac{t_1^*}{Z_m} = \theta_1^T (sI - F_{n-1})^{-1} g_{n-1}
\]

\[
\frac{t_2^*}{Z_m} = \theta_2^* + \theta_2^T (sI - F_{n-1})^{-1} g_{n-1}
\]

TF from $r$ to $y_p$:

\[
= k^* \frac{k_p Z_m}{R_m} = \frac{k_m Z_m}{R_m}
\]
Case (iv) $k_p$ unknown, $Z_p(s)$, $R_p(s)$ unknown - Analytic Part

$$u(t) = \theta^T(t)\omega(t)$$
$$= \theta^*^T\omega(t) + \tilde{\theta}^T(t)\omega(t)$$

$$\theta^* = [k^*, \theta_1^*T, \theta_{20}^*T, \theta_2^*T]^T \quad \omega = [r, \omega_1^T, y_p, \omega_2^T]^T$$
Case (iv) $k_p$ unknown, $Z_p(s)$, $R_p(s)$ unknown - Analytic Part

\[
\begin{align*}
\theta^* &= [k^*, \theta_1^T, \theta_{20}^T, \theta_2^T]^T
\omega &= [r, \omega_1^T, y_p, \omega_2^T]^T
\end{align*}
\]

\[
\begin{align*}
u(t) &= \theta^T(t)\omega(t) \\
&= \theta^*\omega(t) + \tilde{\theta}^T(t)\omega(t)
\end{align*}
\]
Case (iv) $k_p$ unknown, $Z_p(s)$, $R_p(s)$ unknown - Analytic Part

\[
\begin{align*}
u(t) &= \theta^T(t)\omega(t) \\
&= \theta^* T \omega(t) + \tilde{\theta}^T(t)\omega(t) \\
\theta^* &= \begin{bmatrix} k^* , \theta_1^* T , \theta_2^* T , \theta_2^* T \end{bmatrix}^T \\
\omega &= \begin{bmatrix} r , \omega_1^T , y_p , \omega_2^T \end{bmatrix}^T
\end{align*}
\]
Case (iv) Error Model

\[ \dot{k} = -\text{sign}(k_p)e_1r \]
\[ \dot{\theta}_1 = -\text{sign}(k_p)e_1\omega_1 \]
\[ \dot{\theta}_{20} = -\text{sign}(k_p)e_1y_p \]
\[ \dot{\theta}_2 = -\text{sign}(k_p)e_1\omega_2 \]

- \( Z_m(s) \) must have roots in \( C^- \).
- Poles and Zeros must be interlaced.
- \( e, \tilde{\theta} \) bounded; \( \lim_{t \to \infty} e(t) = 0 \). (Barbalet’s Lemma)
- Show that \( \omega \) is bounded.
- and that \( \lim_{t \to \infty} e_1(t) = 0 \).