2.153 Adaptive Control
Lecture 1
Simple Adaptive Systems: Identification

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Adaptive Control: The control of Uncertain Systems
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Adaptive Control (in this Course): The control of Linear Time-invariant Plants with Unknown Parameters
Adaptive Control: A Parametric Framework

- Nonlinear, time-varying, with unknown parameter $\theta$
  \[
  \dot{x} = f(x, u, \theta, t) \quad y = h(x, u, \theta, t)
  \]

- Linear Time-Varying (LTV) with unknown parameter $\theta$
  \[
  \dot{x} = A(\theta, t)x + B(\theta, t)u \quad y = C(\theta, t)x + D(\theta, t)u
  \]

- Linear Time-Invariant (LTI) with unknown parameter $\theta$
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System to be controlled (open-loop): Plant
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System to be controlled (open-loop): Plant
Controlled System (closed-loop): System
Direct and Indirect Adaptive Control

$\theta_p$: Plant parameter - unknown;

$\theta_c$: Control parameter
Direct and Indirect Adaptive Control

\( \theta_p \): Plant parameter - unknown; \( \theta_c \): Control parameter

**Indirect Adaptive Control:** Estimate \( \theta_p \) as \( \hat{\theta}_p \). Compute \( \hat{\theta}_c \) using \( \hat{\theta}_p \).

\[ \theta_p \rightarrow \hat{\theta}_p \rightarrow \hat{\theta}_c \]
Direct and Indirect Adaptive Control

\[ \theta_p: \text{Plant parameter - unknown;} \]
\[ \theta_c: \text{Control parameter} \]

**Indirect Adaptive Control:** Estimate \( \theta_p \) as \( \hat{\theta}_p \). Compute \( \hat{\theta}_c \) using \( \hat{\theta}_p \).

\[ \theta_p \rightarrow \hat{\theta}_p \rightarrow \hat{\theta}_c \]

**Direct Adaptive Control:** Directly estimate \( \theta_c \) as \( \hat{\theta}_c \). Compute the plant estimate \( \hat{\theta}_p \) using \( \hat{\theta}_c \).

\[ \theta_p \rightarrow \theta_c \rightarrow \hat{\theta}_c \]
Identification of a Single Parameter

$\theta$: Unknown, scalar

$y(t) = \theta u(t)$
Identification of a Single Parameter

$\theta$: Unknown, scalar

$y(t) = \theta u(t)$

Identify $\theta$ using measurements $\{u(t), y(t)\}$. 
Identification of a Vector Parameter

\[ y(t) = \theta^T u(t) \]

Identify \( \theta \) using measurements \( \{u(t), y(t)\} \).
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\( y \in \mathbb{R}, \)
Identification of a Vector Parameter

\[ \mathbf{y}(t) = \mathbf{\theta}^T \mathbf{u}(t) \]

\[ \mathbf{y} \in \mathbb{R}, \quad \mathbf{\theta} \in \mathbb{R}^n, \]
Identification of a Vector Parameter

\[ y(t) = \theta^T u(t) \]

\[ y \in \mathbb{R}, \quad \theta \in \mathbb{R}^n, \quad u : \mathbb{R}^+ \to \mathbb{R}^n \]
Identification of a Vector Parameter

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\( y \in \mathbb{R}, \quad \theta \in \mathbb{R}^n, \quad u : \mathbb{R}^+ \rightarrow \mathbb{R}^n \)

Identify \( \theta \) using measurements \( \{u(t), y(t)\} \).
Identification of a Single Parameter - Recursive Scheme

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Identification of a Single Parameter - Recursive Scheme

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\( \theta \): Unknown, scalar  
Identify \( \theta \) as \( \hat{\theta}(t) \) at every instant
Identification of a Vector Parameter - Recursive Scheme

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Identify \( \theta \) as \( \hat{\theta}(t) \) at every instant
Error Model 1

\( \hat{\theta} \): Unknown, \( u(t) \) and \( e(t) \) can be measured at each instant \( t \).
Identification of a Parameter in a Dynamic System

Simplest Transfer Function of a Motor:

\[ V \rightarrow \frac{K}{Js + B} \rightarrow \omega \]

\( V \): Voltage input \hspace{1cm} \( \omega \): Angular Velocity output

\( K, J, B \): Physical parameters

Plant:

\[ \frac{K}{Js + B} = \frac{a_1}{s + \theta_1} \]
Identification of a Parameter in a Dynamic System

Simplest Transfer Function of a Motor:

\[ \frac{V}{\omega} = \frac{K}{Js + B} \]

\[ V : \text{Voltage input} \quad \omega : \text{Angular Velocity output} \]

\[ K, J, B : \text{Physical parameters} \]

Plant:

\[ \frac{K}{Js + B} = \frac{a_1}{s + \theta_1} \]

\[ K, J, B \text{ unknown} \Rightarrow a_1, \theta_1 \text{ unknown} \]
One way of identifying parameters $a_1$ and $\theta_1$

Assume that $a_1$ is known.
One way of identifying parameters $a_1$ and $\theta_1$

Assume that $a_1$ is known. Identify $\theta_1$ as $\hat{\theta}$.
One way of identifying parameters $a_1$ and $\theta_1$

Assume that $a_1$ is known. Identify $\theta_1$ as $\hat{\theta}$. 

$$\tilde{\theta} = \hat{\theta} - \theta_1$$

Plant: $\dot{\omega} = -\theta_1 \omega + u \quad u = a_1 V$
\[ \dot{e} = -\theta_1 e + \tilde{\theta} u \]
An alternate procedure for identifying $\theta_1$:

$$\frac{a_1}{s + \theta_1} = \frac{a_1}{s + \theta_m} \frac{1 + \frac{\theta_m - \theta_1}{s + \theta_m}}{1}$$

$\theta \equiv \theta_1 - \theta_m$
Stability

Behavior near an Equilibrium Point.
Stability

Behavior near an Equilibrium Point.
Consider the following dynamical system

\[ \dot{x}(t) = f(x(t), t) \]
\[ x(t_0) = x_0 \]  

(1)

**Definition: equilibrium point (pg 45)** The state \( x_{eq} \) is an *equilibrium point* of (1) if it satisfies:

\[ f(x_{eq}, t) = 0 \]  

(2)

for all \( t \geq t_0 \).
Stability of LTI Plants

A motivating example: determine the stability of the origin for the following scalar system

\[ \dot{x}(t) = Ax(t) \]

Equilibrium point: \( x = 0 \)
Stability of LTI Plants

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$$\dot{x}(t) = Ax(t)$$

Equilibrium point: \( x = 0 \)

Can determine the stability of the origin by evaluating eigenvalues of \( A \)

$$x(t) = e^{A(t-t_0)}x(t_0)$$

$$A = V\Lambda V^{-1}; \quad V : \text{from eigenvector; } \Lambda : diag(\lambda_i) : \text{from eigenvalues}$$

Stability follows if \( Re(\lambda_i) \leq 0 \)

Asymptotic stability follows if \( Re(\lambda_i) < 0. \)
Stability of LTI Plants

A motivating example: determine the stability of the origin for the following scalar system

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Stability follows if \( Re(\lambda_i) \leq 0 \)
Asymptotic stability follows if \( Re(\lambda_i) < 0 \).

Lyapunov’s methods allow us to determine the stability of an equilibrium for such a system without solving the differential equation!
Lyapunov Stability

For the system

\[ \dot{x} = f(x) \]

Let

(i) \( V(x) > 0, \ \forall x \neq 0, \text{ and } V(0) = 0 \)
(ii) \( \dot{V}(x) = \left( \frac{\partial V}{\partial x} \right)^T f(x) < 0 \)
(ii) \( V(x) \to \infty \text{ as } \|x\| \to \infty \)

Then \( x = 0 \) is asymptotically stable.
Lyapunov Stability

For the system

\[ \dot{x} = f(x) \]

Let

1. \( V(x) > 0, \ \forall x \neq 0, \) and \( V(0) = 0 \)
2. \( \dot{V}(x) = \left( \frac{\partial V}{\partial x} \right)^T f(x) < 0 \)
3. \( V(x) \to \infty \) as \( \|x\| \to \infty \)

Then \( x = 0 \) is asymptotically stable.

If instead of (ii), we have

(ii') \( \dot{V} \leq 0 \)

Then \( x = 0 \) is stable.
Error Model 1

Error Model 1 leads to the following

\[ \dot{x}(t) = A(t)x(t) \quad A(t) = -u(t)u^T(t) \]

Equilibrium point: \( x = 0 \)
Error Model 1

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Equilibrium point: \( x = 0 \)

Choose a quadratic function

\[ V = \frac{1}{2}x^Tx \]

\[ \dot{V} = x^T A(t)x = -x^Tu(t)u^T(t)x = -\left(x^Tu(t)\right)^2 \leq 0 \]
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\[ \Rightarrow \text{stability.} \]
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\( \Rightarrow \) stability.

A later lecture will show that if \( u(t) \) is "persistently exciting", \( x(t) \to 0 \).
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\( \Rightarrow \) stability.
A later lecture will show that if \( u(t) \) is "persistently exciting", \( x(t) \rightarrow 0 \). We therefore conclude that error model 1 leads to a stable parameter estimation. Asymptotic stability will be shown later.