Adaptive Control of $n$th order plants - with single input

Plant: $\dot{X}_p = A_p X_p + b_p u,$
Adaptive Control of $n$th order plants - with single input

Plant: $\dot{X}_p = A_p X_p + b_p u, A_p \in \mathbb{R}^{n \times n}, b_p \in \mathbb{R}^n, u \in \mathbb{R}$

Controller: $u = \theta^T_c X_p + k_c r$

Matching conditions: $A_p + b_p \theta^*_T = A_m; b_p k_* = b_m$

Reference Model: $\dot{X}_m = A_m X_m + b_m r$

Solution: $\theta^*_c = \theta^*_*, k^*_c = k^*_*$

Choose Controller: $u(t) = \theta^T_c(t) X_p + k_c(t) r$

Closed-loop: $\dot{X}_p = [A_p + b_p \theta^*_T] X_p + b_p (k_* + \tilde{k}) r = A_m X_p + b_p \tilde{\theta}^*_T X_p + b_p \tilde{k} r + b_m r$
Adaptive Control of $n$th order plants - with single input

Plant: $\dot{X}_p = A_p X_p + b_p u, A_p \in \mathbb{R}^{n \times n}, b_p \in \mathbb{R}^n, u \in \mathbb{R}$

Controller: $u = \theta^T X_p + k_c r$

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Adaptive Control of $n$th order plants - with single input

Plant: $\dot{X}_p = A_p X_p + b_p u, A_p \in \mathbb{R}^{n \times n}, b_p \in \mathbb{R}^n, u \in \mathbb{R}$

Controller: $u = \theta_c^T X_p + k_c r$

Closed-loop: $\dot{X}_p = [A_p + b_p \theta_c^T] X_p + b_p k_c r$

Matching conditions: $A_p + b_p \theta^* T = A_m; b_p k^* = b_m$
Adaptive Control of $n$th order plants - with single input

Plant: \[ \dot{X}_p = A_p X_p + b_p u, \quad A_p \in \mathbb{R}^{n \times n}, \quad b_p \in \mathbb{R}^n, \quad u \in \mathbb{R} \]

Controller: \[ u = \theta_c^T X_p + k_c r \]

Closed-loop: \[ \dot{X}_p = \left[ A_p + b_p \theta_c^T \right] X_p + b_p k_c r \]

Matching conditions: \[ A_p + b_p \theta^* T = A_m; \quad b_p k^* = b_m \]

Reference Model \[ \dot{X}_m = A_m X_m + b_m r \]
Adaptive Control of $n$th order plants - with single input

Plant: $\dot{X}_p = A_p X_p + b_p u, A_p \in \mathbb{R}^{n \times n}, b_p \in \mathbb{R}^n, u \in \mathbb{R}$

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Closed-loop: $\dot{X}_p = [A_p + b_p \theta_c^T] X_p + b_p k_c r$

Matching conditions: $A_p + b_p \theta^* T = A_m; b_p k^* = b_m$

Reference Model $\dot{X}_m = A_m X_m + b_m r$

Solution: $\theta_c = \theta^*, k_c = k^*$
Adaptive Control of \( n \)th order plants - with single input

Plant: \[ \dot{X}_p = A_p X_p + b_p u, \ A_p \in \mathbb{R}^{n \times n}, \ b_p \in \mathbb{R}^n, \ u \in \mathbb{R} \]

Controller: \[ u = \theta_c^T X_p + k_c r \]

Closed-loop: \[ \dot{X}_p = [A_p + b_p \theta_c^T] X_p + b_p k_c r \]

Matching conditions: \[ A_p + b_p \theta^*^T = A_m; \ b_p k^* = b_m \]

Reference Model: \[ \dot{X}_m = A_m X_m + b_m r \]

Solution: \[ \theta_c = \theta^*, \ k_c = k^* \]

\( A_p, b_p \) unknown \( \implies \theta^*, k^* \) unknown

Choose Controller: \[ u = \theta^T(t) X_p + k(t) r \]
Adaptive Control of \( n \)th order plants - with single input

Plant: \( \dot{X}_p = A_p X_p + b_p u, A_p \in \mathbb{R}^{n \times n}, b_p \in \mathbb{R}^n, u \in \mathbb{R} \)

Controller: \( u = \theta^T_c X_p + k_c r \)

Closed-loop: \( \dot{X}_p = [A_p + b_p \theta^T_c] X_p + b_p k_c r \)

Matching conditions: \( A_p + b_p \theta^*^T = A_m; b_p k^* = b_m \)

Reference Model
\( \dot{X}_m = A_m X_m + b_m r \)

Solution: \( \theta_c = \theta^*, k_c = k^* \)

\( A_p, b_p \) unknown \( \implies \theta^*, k^* \) unknown

Choose Controller: \( u = \theta^T(t) X_p + k(t) r \)

Closed-loop: \( \dot{X}_p = [A_p + b_p \theta^T(t)] X_p + b_p (k^* + \tilde{k}) r \)
Adaptive Control of \( n \)th order plants - with single input

**Plant:** \( \dot{X}_p = A_p X_p + b_p u, \ A_p \in \mathbb{R}^{n \times n}, \ b_p \in \mathbb{R}^n, \ u \in \mathbb{R} \)

**Controller:** \( u = \theta^T_c X_p + k_c r \)

**Closed-loop:** \( \dot{X}_p = [A_p + b_p \theta^T_c] X_p + b_p k_c r \)

**Matching conditions:** \( A_p + b_p \theta^* T = A_m; \ b_p k^* = b_m \)

**Reference Model**

\( \dot{X}_m = A_m X_m + b_m r \)

**Solution:** \( \theta_c = \theta^*, \ k_c = k^* \)

\( A_p, b_p \) unknown \( \implies \) \( \theta^*, k^* \) unknown

**Choose Controller:** \( u = \theta^T(t) X_p + k(t) r \)

**Closed-loop:** \( \dot{X}_p = [A_p + b_p \theta^T(t)] X_p + b_p (k^* + \tilde{k}) r \)

\( = A_m X_p + b_p \left( \tilde{\theta}^T X_p + \tilde{k} r \right) + b_m r \)
Error Model 2 and Stability Analysis

Error equation:
\[
\dot{e} = A_m e + b_p \left( \tilde{\theta}^T X_p + \tilde{k} r \right)
\]
Error Model 2 and Stability Analysis

Error equation:  \[
\dot{e} = A_m e + b_p \left( \tilde{\theta}^T X_p + \tilde{k} r \right)
\]

\[
V = e^T P e + |k^*| \left( \tilde{\theta}^T \tilde{\theta} + \tilde{k}^2 \right)
\]
Error Model 2 and Stability Analysis

Error equation:
\[ \dot{e} = A_m e + b_p \left( \tilde{\theta}^T X_p + \tilde{k} r \right) \]
\[ V = e^T P e + |k^*| \left( \tilde{\theta}^T \tilde{\theta} + \tilde{k}^2 \right) \]
\[ \dot{V} = e^T [A_m^T P + P A_m] e + 2 e^T P b_p \tilde{\theta}^T X_p + 2 |k^*| \tilde{\theta}^T \dot{\tilde{\theta}} \]
\[ + 2 e^T P b_p \tilde{k} r + 2 |k^*| \tilde{k} \dot{\tilde{k}} \]
\[ = -e^T Q e \]

if \( \dot{\tilde{\theta}} = -\text{sign}(k^*) e^T P b_m X_p \), \( \dot{\tilde{k}} = -\text{sign}(k^*) e^T P b_m r \)

\( \Rightarrow e(t), \tilde{\theta}(t), \text{ and } \tilde{k}(t) \) are bounded for all \( t \geq t_0 \)
Error Model 2 and Stability Analysis

\[ [X_p] \rightarrow \begin{bmatrix} \tilde{\theta} \\ \tilde{k} \end{bmatrix}^T \rightarrow (sI - A_m)^{-1}b_p \rightarrow e \]

Error equation:
\[ \dot{e} = A_m e + b_p \left( \tilde{\theta}^T X_p + \tilde{k} r \right) \]

\[ V = e^T P e + |k^*| \left( \tilde{\theta}^T \tilde{\theta} + \tilde{k}^2 \right) \]

\[ \dot{V} = e^T \left[ A_m^T P + P A_m \right] e + 2e^T P b_p \tilde{\theta}^T X_p + 2|k^*| \tilde{\theta}^T \dot{\tilde{\theta}} + 2e^T P b_p \tilde{k} r + 2|k^*| \tilde{k} \dot{\tilde{k}} \]

\[ \dot{V} = -e^T Q e \]

if \[ \tilde{\theta} = -sign(k^*) e^T P b_m X_p, \tilde{k}^* = -sign(k^*) e^T P b_m r \]

\[ \Rightarrow e(t), \tilde{\theta}(t), \text{ and } \tilde{k}(t) \text{ are bounded for all } t \geq t_0 \]

\[ \lim_{t \to \infty} e(t) = 0 \text{ from Barbalat’s Lemma} \]
LTI System: \( \dot{x} = A_m x \)

Theorem: Given \( Q = Q^T > 0 \), there exists \( P = P^T > 0 \) that solves
\[
A^T_m P + PA_m = -Q
\]
if and only if \( A_m \) is a Hurwitz matrix.

Main implication:
\[
\dot{V} = x^T P \dot{x} = x^T \left[ A^T_m P + PA_m \right] x = -x^T Q x < 0
\]
Lyapunov functions and Linear Time-invariant Systems

LTI System: \[ \dot{x} = A_m x \]

Theorem: Given \( Q = Q^T > 0 \), there exists \( P = P^T > 0 \) that solves
\[
A_m^T P + PA_m = -Q
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Lyapunov functions and Linear Time-invariant Systems

LTI System: \[ \dot{x} = A_m x \]

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\[
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\]
if and only if \( A_m \) is a Hurwitz matrix.

Main implication:
\[
V = x^T P x
\]
LTI System: \[ \dot{x} = A_m x \]

Theorem: Given \( Q = Q^T > 0 \), there exists \( P = P^T > 0 \) that solves

\[ A_m^T P + PA_m = -Q \]

if and only if \( A_m \) is a Hurwitz matrix.

Main implication:

\[ \begin{align*}
V &= x^T P x \\
\dot{V} &= x^T [A_m^T P + PA_m] x
\end{align*} \]
Lyapunov functions and Linear Time-invariant Systems

LTI System: \( \dot{x} = A_m x \)

Theorem: Given \( Q = Q^T > 0 \), there exists \( P = P^T > 0 \) that solves
\[
A_m^T P + PA_m = -Q
\]
if and only if \( A_m \) is a Hurwitz matrix.

Main implication:
\[
V = x^T P x \\
\dot{V} = x^T [A_m^T P + PA_m] x \\
= -x^T Q x < 0
\]
Overall Adaptive System - single input

Assumption: $\theta^*$ and $k^*$ exist such that

$$b_p k^* = b_m$$
$$A_p + b_p \theta^{*\top} = A_m$$

$$\dot{\theta} = -\text{sign}(k^*) e^\top P b_m X_p$$
$$\dot{k} = -\text{sign}(k^*) e^\top P b_m r$$

$\Rightarrow$ Stability, $e(t) \to 0$
Adaptive Control of $n$th order plants - with multiple inputs

Plant: $\dot{X}_p = A_pX_p + B_pu,$

$A_p \in \mathbb{R}^{n \times n}, \quad B_p \in \mathbb{R}^{n \times m}, \quad u \in \mathbb{R}^m$
Adaptive Control of $n$th order plants - with multiple inputs

Plant: \[ \dot{X}_p = A_p X_p + B_p u, \]
\[ A_p \in \mathbb{R}^{n \times n}, \quad B_p \in \mathbb{R}^{n \times m}, \quad u \in \mathbb{R}^m \]

Controller: \[ u = \Theta_{Ac} X_p + \Theta_{Bc} r \]

Closed-loop: \[ \dot{X}_p = [A_p + B_p \Theta_{Ac}] X_p + B_p \Theta_{Bc} r \]
Adaptive Control of $n$th order plants - with multiple inputs

Plant: \[ \dot{X}_p = A_p X_p + B_p u, \]
\[ A_p \in \mathbb{R}^{n \times n}, \quad B_p \in \mathbb{R}^{n \times m}, \quad u \in \mathbb{R}^m \]

Controller: \[ u = \Theta_{Ac} X_p + \Theta_{Bc} r \]

Closed-loop: \[ \dot{X}_p = [A_p + B_p \Theta_{Ac}] X_p + B_p \Theta_{Bc} r \]

Matching conditions: \[ A_p + B_p \Theta^*_A = A_m; \quad B_p \Theta^*_B = B_m \]
Adaptive Control of $n$th order plants - with multiple inputs

Plant: \[
\dot{X}_p = A_p X_p + B_p u,
\]
\[
A_p \in \mathbb{R}^{n \times n}, \quad B_p \in \mathbb{R}^{n \times m}, \quad u \in \mathbb{R}^m
\]

Controller: \[
\dot{X}_p = \Theta_{Ac} X_p + \Theta_{Bc} r
\]

Closed-loop: \[
\dot{X}_p = [A_p + B_p \Theta_{Ac}] X_p + B_p \Theta_{Bc} r
\]

Matching conditions: \[
A_p + B_p \Theta_A^* = A_m; \quad B_p \Theta_B^* = B_m
\]

Reference Model \[
\dot{X}_m = A_m X_m + B_m r
\]
Adaptive Control of $n$th order plants - with multiple inputs

**Plant:** \( \dot{X}_p = A_p X_p + B_p u, \)
\( A_p \in \mathbb{R}^{n \times n}, \quad B_p \in \mathbb{R}^{n \times m}, \quad u \in \mathbb{R}^m \)

**Controller:** \( u = \Theta_{Ac} X_p + \Theta_{Bc} r \)

**Closed-loop:** \( \dot{X}_p = [A_p + B_p \Theta_{Ac}] X_p + B_p \Theta_{Bc} r \)

**Matching conditions:**
\( A_p + B_p \Theta_A^* = A_m; \quad B_p \Theta_B^* = B_m \)

**Reference Model**
\( \dot{X}_m = A_m X_m + B_m r \)

**Solution:** \( \Theta_{Ac} = \Theta_A^*, \quad \Theta_{Bc} = \Theta_B^* \)

(aanna@mit.edu)

Jan 22, 1400-1500
Adaptive Control of $n$th order plants - with multiple inputs

Plant: \[ \dot{X}_p = A_p X_p + B_p u, \]
\[ A_p \in \mathbb{R}^{n \times n}, \quad B_p \in \mathbb{R}^{n \times m}, \quad u \in \mathbb{R}^m \]

Controller: \[ u = \Theta_{Ac} X_p + \Theta_{Bc} r \]

Closed-loop: \[ \dot{X}_p = [A_p + B_p \Theta_{Ac}] X_p + B_p \Theta_{Bc} r \]

Matching conditions: \[ A_p + B_p \Theta_A^* = A_m; \quad B_p \Theta_B^* = B_m \]

Reference Model \[ \dot{X}_m = A_m X_m + B_m r \]

Solution: \[ \Theta_{Ac} = \Theta_A^*, \quad \Theta_{Bc} = \Theta_B^* \]
\[ \Theta_A^* \in \mathbb{R}^{m \times n}, \quad \Theta_B^* \in \mathbb{R}^{m \times m} \]

$A_p, B_p$ unknown \[ \implies \Theta_A^*, \Theta_B^* \text{ unknown} \]
Adaptive Control of \( n \)th order plants - with multiple inputs; \( B_p \) known

Plant:
\[
\dot{X}_p = A_p X_p + B_p u
\]

Choose Controller:
\[
u = \Theta_A(t) X_p + \Theta_B^* r
\]
Adaptive Control of \( n \)th order plants - with multiple inputs; \( B_p \) known

Plant: \( \dot{X}_p = A_p X_p + B_p u \)

Choose Controller: \( u = \Theta_A(t) X_p + \Theta_B^* r \)

Closed-loop: \( \dot{X}_p = [A_p + B_p \Theta_A(t)] X_p + B_p \Theta_B^* r \)
Adaptive Control of $n$th order plants - with multiple inputs; $B_p$ known

Plant: \[ \dot{X}_p = A_pX_p + B_pu \]

Choose Controller: \[ u = \Theta_A(t)X_p + \Theta^*_B r \]

Closed-loop: \[ \dot{X}_p = [A_p + B_p\Theta_A(t)]X_p + B_p\Theta^*_B r \]

\[ A_p + B_p\Theta^*_A = A_m; \quad B_p\Theta^*_B = B_m, \quad \tilde{\Theta}_A = \Theta_A - \Theta^*_A \]
Adaptive Control of \( n \)th order plants - with multiple inputs; \( B_p \) known

Plant: \[ \dot{X}_p = A_p X_p + B_p u \]

Choose Controller: \[ u = \Theta_A(t) X_p + \Theta_B^* r \]

Closed-loop: \[ \dot{X}_p = [A_p + B_p \Theta_A(t)] X_p + B_p \Theta_B^* r \]

\[
\begin{align*}
A_p + B_p \Theta_A^* &= A_m; & B_p \Theta_B^* &= B_m; & \tilde{\Theta}_A &= \Theta_A - \Theta_A^* \\
\dot{X}_p &= A_m X_p + B_p \tilde{\Theta}_A X_p + B_m r
\end{align*}
\]
Adaptive Control of $n$th order plants - with multiple inputs; $B_p$ known

Plant: $\dot{X}_p = A_p X_p + B_p u$

Choose Controller: $u = \Theta_A(t) X_p + \Theta_B^* r$

Closed-loop: $\dot{X}_p = [A_p + B_p \Theta_A(t)] X_p + B_p \Theta_B^* r$

$A_p + B_p \Theta_A^* = A_m$; $B_p \Theta_B^* = B_m$, $\tilde{\Theta}_A = \Theta_A - \Theta_A^*$

$\dot{X}_p = A_m X_p + B_p \tilde{\Theta}_A X_p + B_m r$

Reference Model $\dot{X}_m = A_m X_m + B_m r$
Adaptive Control of \( n \)th order plants - with multiple inputs; \( B_p \) known

Plant: \[ \dot{X}_p = A_p X_p + B_p u \]

Choose Controller: \[ u = \Theta_A(t) X_p + \Theta_B^* r \]

Closed-loop: \[ \dot{X}_p = [A_p + B_p \Theta_A(t)] X_p + B_p \Theta_B^* r \]

\[ A_p + B_p \Theta_A^* = A_m; \quad B_p \Theta_B^* = B_m, \quad \tilde{\Theta}_A = \Theta_A - \Theta_A^* \]

\[ \dot{X}_p = A_m X_p + B_p \tilde{\Theta}_A X_p + B_m r \]

Reference Model \[ \dot{X}_m = A_m X_m + B_m r \]

Error Model \[ \dot{e} = A_m e + B_p \tilde{\Theta}_A X_p \]

Reading: 3.4, 2.4.4
Error Model 2 and Stability Analysis

Error equation: \( \dot{e} = A_m e + B_p \tilde{\Theta}_A X_p \)
Error Model 2 and Stability Analysis

Error equation: \[ \dot{e} = A_m e + B_p \tilde{\Theta}_A X_p \]

\[ \tilde{\Theta}_A \in \mathbb{R}^{m \times n} \]
Error Model 2 and Stability Analysis

Error equation: \[ \dot{e} = A_m e + B_p \tilde{\Theta}_A X_p \]
\[ \tilde{\Theta}_A \in \mathbb{R}^{m \times n} \]

Use of Trace operator: converts matrices to scalars.
Error Model 2 and Stability Analysis

Error equation: \[
\dot{e} = A_m e + B_p \tilde{\Theta}_A X_p
\]

\[
\tilde{\Theta}_A \in \mathbb{R}^{m \times n}
\]

Use of Trace operator: converts matrices to scalars.

\[
\text{Trace}(ab^T) = b^T a, \quad a, b \in \mathbb{R}^n
\]
Error equation: \[ \dot{e} = A_m e + B_p \tilde{\Theta}_A X_p \]
\[ \tilde{\Theta}_A \in \mathbb{R}^{m \times n} \]

Use of $Trace$ operator: converts matrices to scalars.

\[ Trace(ab^T) = b^T a, \quad a, b \in \mathbb{R}^n \]

\[ V = e^T Pe + Tr \left( \tilde{\Theta}_A^T \tilde{\Theta}_A \right) \]
Error Model 2 and Stability Analysis

Error equation: \( \dot{e} = A_m e + B_p \tilde{\Theta}_A X_p \)
\( \tilde{\Theta}_A \in \mathbb{R}^{m \times n} \)

Use of Trace operator: converts matrices to scalars.

\[ \text{Trace}(ab^T) = b^T a, \quad a, b \in \mathbb{R}^n \]

\[ V = e^T P e + \text{Tr} \left( \tilde{\Theta}_A^T \tilde{\Theta}_A \right) \]

\[ \dot{V} = e^T [A_m^T P + PA_m] e + 2 e^T P B_p \tilde{\Theta}_A X_p + 2 \text{Tr} \left( \tilde{\Theta}_A^T \dot{\tilde{\Theta}}_A \right) \]

Choose \( \dot{\tilde{\Theta}}_A = -B_p^T P e X_p^T \)

\[ \dot{V} = -e^T Q e \leq 0 \]

\[ \lim_{t \to \infty} e(t) = 0 \text{ from Barbalat's Lemma} \]
Error Model 2 and Stability Analysis

Error equation: \( \dot{e} = A_m e + B_p \tilde{\Theta}_A X_p \)

\( \tilde{\Theta}_A \in \mathbb{R}^{m \times n} \)

Use of Trace operator: converts matrices to scalars.

\[
\text{Trace}(ab^T) = b^T a, \quad a, b \in \mathbb{R}^n
\]

\[
V = e^T P e + \text{Tr} \left( \tilde{\Theta}_A^T \tilde{\Theta}_A \right)
\]

\[
\dot{V} = e^T [A_m^T P + P A_m] e + 2 e^T P B_p \tilde{\Theta}_A X_p + 2 \text{Tr} \left( \tilde{\Theta}_A^T \dot{\tilde{\Theta}}_A \right)
\]

Choose \( \dot{\tilde{\Theta}}_A = -B_p^T P e X_p^T \)

\[
\dot{V} = -e^T Q e \leq 0
\]

\( \Rightarrow e(t) \) and \( \tilde{\Theta}_A(t) \) are bounded for all \( t \geq t_0 \)
Error equation: \[ \dot{e} = A_m e + B_p \tilde{\Theta}_A X_p \]
\[ \tilde{\Theta}_A \in \mathbb{R}^{m \times n} \]

Use of Trace operator: converts matrices to scalars.

\[ \text{Trace}(ab^T) = b^T a, \quad a, b \in \mathbb{R}^n \]

\[ V = e^T P e + \text{Tr} \left( \tilde{\Theta}_A^T \tilde{\Theta}_A \right) \]

\[ \dot{V} = e^T [A_m^T P + PA_m] e + 2 e^T P B_p \tilde{\Theta}_A X_p + 2 \text{Tr} \left( \tilde{\Theta}_A^T \dot{\tilde{\Theta}}_A \right) \]

Choose \( \dot{\tilde{\Theta}}_A = -B_p^T P e X_p^T \)

\[ \dot{V} = -e^T Q e \leq 0 \]

\[ \Rightarrow e(t) \quad \text{and} \quad \tilde{\Theta}_A(t) \quad \text{are bounded for all} \quad t \geq t_0 \]

\[ \lim_{t \to \infty} e(t) = 0 \quad \text{from Barbalat’s Lemma} \]