2.153 Adaptive Control
Fall 2019
Lecture 21: Adaptive Control: $n^* \geq 2$

Anuradha Annaswamy

aanna@mit.edu

November 17, 2019
Adaptive Control using Output Feedback

Goal: Choose $u$ so that $e_1(t)$ tends to zero asymptotically.
Adaptive Control using Output Feedback

Goal: Choose $u$ so that $e_1(t)$ tends to zero asymptotically. $k_p$ and coefficients of $R_p(s)$ and $Z_p(s)$ are unknown.
Adaptive Control using Output Feedback

Goal: Choose $u$ so that $e_1(t)$ tends to zero asymptotically. $k_p$ and coefficients of $R_p(s)$ and $Z_p(s)$ are unknown. $R_p(s)$ is monic and of degree $n$. 
Adaptive Control using Output Feedback

Goal: Choose $u$ so that $e_1(t)$ tends to zero asymptotically. $k_p$ and coefficients of $R_p(s)$ and $Z_p(s)$ are unknown. $R_p(s)$ is monic and of degree $n$. $Z_p(s)$ is monic and of degree $m \leq n - 2$. 
Adaptive Control using Output Feedback

Goal: Choose $u$ so that $e_1(t)$ tends to zero asymptotically. $k_p$ and coefficients of $R_p(s)$ and $Z_p(s)$ are unknown. $R_p(s)$ is monic and of degree $n$. $Z_p(s)$ is monic and of degree $m \leq n - 2$.

$R_m(s)$ is monic, Hurwitz, and of degree $n$. 
Adaptive Control using Output Feedback

Goal: Choose $u$ so that $e_1(t)$ tends to zero asymptotically. $k_p$ and coefficients of $R_p(s)$ and $Z_p(s)$ are unknown. $R_p(s)$ is monic and of degree $n$. $Z_p(s)$ is monic and of degree $m \leq n - 2$.

$R_m(s)$ is monic, Hurwitz, and of degree $n$. $Z_m(s)$ is monic and of degree $m \leq n - 2$. 
Goal: Choose $u$ so that $e_1(t)$ tends to zero asymptotically. $k_p$ and coefficients of $R_p(s)$ and $Z_p(s)$ are unknown. $R_p(s)$ is monic and of degree $n$. $Z_p(s)$ is monic and of degree $m \leq n - 2$.

$R_m(s)$ is monic, Hurwitz, and of degree $n$. $Z_m(s)$ is monic and of degree $m \leq n - 2$.

Assumptions:

- sign $k_p$ known
- $Z_p(s)$ has roots in $C^-$.
Algebraic Part

\[ k_c + \frac{R_pZ_p}{R_p} \]

Plant

\[ \frac{t_1}{Z_m} \]

\[ \frac{t_2}{Z_m} \]

Not enough degrees of freedom in \( t_2^* (s) \). Need Bezout Identity.

Replace \( Z_m(s) \) by \( Z'_m(s) = \lambda(s) Z_m(s) \)

\[ \deg(\lambda) = n - 1 - m \]

New design:

\[ t_1^* Z'_m = \theta^* T_1 (sI - F_n - 1) - 1 g_n - 1 \]

\[ t_2^* Z'_m = \theta^* T_2 (sI - F_n - 1) - 1 g_n - 1 \]
Algebraic Part

\[
\frac{t_1^*}{Z_m} = \theta_1^* T (sI - F_{n-1})^{-1} g_{n-1}
\]
Algebraic Part

\[ t_1^* = \theta_1^* (sI - F_{n-1})^{-1} g_{n-1} \]
\[ t_2^* = \theta_2^* + \theta_2^* (sI - F_{n-1})^{-1} g_{n-1} \]
Algebraic Part

\[ \frac{t_1^*}{Z_m} = \theta_1^* T (sI - F_{n-1})^{-1} g_{n-1} \]

\[ \frac{t_2^*}{Z_m} = \theta_2^* + \theta_2^* T (sI - F_{n-1})^{-1} g_{n-1} \]

\[ Z_m(s) - t_1^*(s) = Z_p(s), \quad k^* = \frac{k_m}{k_p} \]

\[ R_p(s) - k_p t_2^*(s) = R_m(s) \]
Algebraic Part

\[
\begin{align*}
t_1^* &= \theta_1^* T (sI - F_{n-1})^{-1} g_{n-1} \\
t_2^* &= \theta_2^* + \theta_2^* T (sI - F_{n-1})^{-1} g_{n-1}
\end{align*}
\]

\[
Z_m(s) - t_1^*(s) = Z_p(s), \quad k^* = \frac{k_m}{k_p}
\]

\[
R_p(s) - k_p t_2^*(s) = R_m(s) \times
\]

Not enough degrees of freedom in \( t_2^*(s) \)! Need Bezout Identity.
Algebraic Part

\[
\frac{t_1^*}{Z_m} = \theta_1^* T (sI - F_{n-1})^{-1} g_{n-1} \quad \frac{t_2^*}{Z_{m}} = \theta_2^* + \theta_2^* T (sI - F_{n-1})^{-1} g_{n-1}
\]

\[
Z_m(s) - t_1^*(s) = Z_p(s), \quad k^* = \frac{k_m}{k_p}
\]

\[
R_p(s) - k_p t_2^*(s) = R_m(s) \times
\]

Not enough degrees of freedom in \( t_2^*(s) \)! Need Bezout Identity.
Replace \( Z_m(s) \) by \( Z'_m(s) = \lambda(s)Z_m(s) \)
Algebraic Part

\[
\begin{align*}
\frac{t_1^*}{Z_m} &= \theta_1^* T (sI - F_{n-1})^{-1} g_{n-1} \\
\frac{t_2^*}{Z_m} &= \theta_2^* + \theta_2^* T (sI - F_{n-1})^{-1} g_{n-1}
\end{align*}
\]

\[
Z_m(s) - t_1^*(s) = Z_p(s), \quad k^* = \frac{k_m}{k_p}
\]

\[
R_p(s) - k_p t_2^*(s) = R_m(s) \times
\]

Not enough degrees of freedom in \( t_2^*(s) \)! Need Bezout Identity.
Replace \( Z_m(s) \) by \( Z'_m(s) = \lambda(s)Z_m(s) \) \( \deg(\lambda) = n - 1 - m \)

New design:
\[
\frac{t_1^*}{Z'_m} = \theta_1^* T (sI - F_{n-1})^{-1} g_{n-1}
\]
Algebraic Part

\[
t_1^* = \theta_1^* T (sI - F_{n-1})^{-1} g_{n-1} \\
Z_m(s) - t_1^*(s) = Z_p(s), k^* = \frac{k_m}{k_p} \\
R_p(s) - k_p t_2^*(s) = R_m(s) \times
\]

Not enough degrees of freedom in \( t_2^*(s) \)! Need Bezout Identity.

Replace \( Z_m(s) \) by \( Z'_m(s) = \lambda(s)Z_m(s) \) \( \text{deg}(\lambda) = n - 1 - m \)

New design:

\[
t_1^* = \theta_1^* T (sI - F_{n-1})^{-1} g_{n-1} \\
t_2^* = \theta_2^* + \theta_2^* T (sI - F_{n-1})^{-1} g_{n-1}
\]
TF from $r$ to $y_p = k_c \cdot \frac{1}{1 - \frac{t_1}{Z_m'}} \cdot \frac{k_p Z_p}{R_p} \cdot \frac{1}{1 - \frac{t_1}{Z_m'}} \cdot \frac{k_p Z_p}{R_p} \cdot \frac{t_2}{Z_m'}
TF from $r$ to $y_p = k_c \cdot \frac{1}{1 - \frac{t_1}{Z_m'}} \cdot \frac{k_p Z_p}{R_p} \cdot \frac{Z_m'}{Z_m - t_1} \cdot \frac{k_p Z_p}{R_p} \cdot \frac{t_2}{Z_m'}
Algebraic part (contd.)

TF from $r$ to $y_p$  

\[
\begin{align*}
TF \text{ from } r \text{ to } y_p &= \ k_c \cdot \frac{1}{1 - \frac{t_1}{Z'_m}} \cdot \frac{k_p Z_p}{R_p} \\
&= \ k_c \cdot \frac{k_p Z_p}{1 - \frac{1}{1 - \frac{t_1}{Z'_m}}} \cdot \frac{t_2}{Z'_m} \\
&= \ k_c \cdot \frac{k_p Z_p}{Z'_m - t_1} \cdot \frac{t_2}{Z'_m} \\
&= \ k_c k_p Z'_m Z_p \frac{k_m Z_m \lambda(s) Z_p(s)}{R_m \lambda(s) Z_p(s)} \\
&= \ k_m Z_m \frac{R_m \lambda(s) Z_p(s)}{R_m} \\
&= \ k_m Z_m \frac{R_m \lambda(s) Z_p(s)}{R_m}
\end{align*}
\]

(from Bezout Identity)
Algebraic part (contd.)

TF from $r$ to $y_p = k_c \cdot \frac{1}{1 - \frac{t_1}{Z'_m}} \cdot \frac{k_p Z_p}{R_p}

\quad = \quad \frac{k_c k_p Z'_m Z_p}{R_p(Z'_m - t_1) - (k_p Z_p)t_2(s)}

= \quad \frac{k_m Z_m \lambda(s) Z_p(s)}{R_m \lambda(s) Z_p(s)}

(\text{from Bezout Identity})

= \quad \frac{k_m Z_m}{R_m} = W_m(s) \quad \text{- Reference Model}
Bezout Identity

\[ R_p(s)(Z'_m(s) - t_1(s)) - (kpZ_p(s))t_2(s) = R_m(s)\lambda(s)Z_p(s) \]
Bezout Identity

\[
\text{Need } R_p(s)(Z'_m(s) - t_1(s)) - (k_pZ_p(s))t_2(s) = R_m(s)\lambda(s)Z_p(s)
\]
Bezout Identity

Need \( R_p(s)(Z'_m(s) - t_1(s)) - (k_pZ_p(s))t_2(s) = R_m(s)\lambda(s)Z_p(s) \)

\[ n + n - 1 \]
Bezout Identity

Need \( R_p(s)(Z'_m(s) - t_1(s)) - (k_pZ_p(s))t_2(s) = R_m(s)\lambda(s)Z_p(s) \)

\[ n + n - 1 \quad n + \]
Bezout Identity

\[ R_p(s)(Z'_m(s) - t_1(s)) - (k_pZ_p(s))t_2(s) = R_m(s)\lambda(s)Z_p(s) \]

\[ n + n - 1 \quad n + \deg(\lambda) + m \]
Bezout Identity

Need $R_p(s)(Z'_m(s) - t_1(s)) - (k_pZ_p(s))t_2(s) = R_m(s)\lambda(s)Z_p(s)$

$n + n - 1 = n + \text{deg}(\lambda) + m = 2n - 1$

Bezout Identity:
$P(s)$: Polynomial of degree $m \leq n - 1$
Bezout Identity

\[ R_p(s)(Z'_m(s) - t_1(s)) - (k_pZ_p(s))t_2(s) = R_m(s)\lambda(s)Z_p(s) \]
\[ n + n - 1 = n + \text{deg}(\lambda) + m = 2n - 1 \]

Bezout Identity:

\[ P(s): \text{Polynomial of degree } m \leq n - 1 \]
\[ Q(s): \text{Polynomial of degree } n \]
Bezout Identity

\[ R_p(s)(Z_m'(s) - t_1(s)) - (k_pZ_p(s))t_2(s) = R_m(s)\lambda(s)Z_p(s) \]
\[ n + n - 1 \quad n + \text{deg}(\lambda) + m = 2n - 1 \]

Bezout Identity:

\( P(s) \): Polynomial of degree \( m \leq n - 1 \)
\( Q(s) \): Polynomial of degree \( n \)
\( M(s) \): Monic polynomial of degree \( n - 1 \)
Bezout Identity

\[ R_p(s)(Z'_m(s) - t_1(s)) - (k_pZ_p(s))t_2(s) = R_m(s)\lambda(s)Z_p(s) \]
\[ n + n - 1 = n + \deg(\lambda) + m = 2n - 1 \]

Bezout Identity:
\( P(s) \): Polynomial of degree \( m \leq n - 1 \)
\( Q(s) \): Polynomial of degree \( n \)
\( M(s) \): Monic polynomial of degree \( n - 1 \)
\( N(s) \): Polynomial of degree \( n - 1 \)
Bezout Identity

\[ R_p(s)(Z'_m(s) - t_1(s)) - (k_pZ_p(s))t_2(s) = R_m(s)\lambda(s)Z_p(s) \]

\[ n + n - 1 = n + \text{deg}(\lambda) + m = 2n - 1 \]

Bezout Identity:
\( P(s) \): Polynomial of degree \( m \leq n - 1 \)
\( Q(s) \): Polynomial of degree \( n \)
\( M(s) \): Monic polynomial of degree \( n - 1 \)
\( N(s) \): Polynomial of degree \( n - 1 \)
\( Q^*(s) \): Monic polynomial of degree \( 2n - 1 \)
Bezout Identity

Need \( R_p(s)(Z'_m(s) - t_1(s)) - (k_pZ_p(s))t_2(s) = R_m(s)\lambda(s)Z_p(s) \)

\[ n + n - 1 \quad n + \text{deg}(\lambda) + m = 2n - 1 \]

Bezout Identity:

\( P(s) \): Polynomial of degree \( m \leq n - 1 \)
\( Q(s) \): Polynomial of degree \( n \)
\( M(s) \): Monic polynomial of degree \( n - 1 \)
\( N(s) \): Polynomial of degree \( n - 1 \)
\( Q^*(s) \): Monic polynomial of degree \( 2n - 1 \)

Given \( P(s) \) and \( Q(s) \) that are coprime, and \( Q^*(s) \), there exist \( M(s) \) and \( N(s) \) such that

\[ Q(s)M(s) + P(s)N(s) = Q^*(s) \]
Bezout Identity

Need $R_p(s)(Z'_m(s) - t_1(s)) - (k_pZ_p(s))t_2(s) = R_m(s)\lambda(s)Z_p(s)$

$$n + n - 1 \quad n + \deg(\lambda) + m = 2n - 1$$

Bezout Identity:
$P(s)$: Polynomial of degree $m \leq n - 1$
$Q(s)$: Polynomial of degree $n$
$M(s)$: Monic polynomial of degree $n - 1$
$N(s)$: Polynomial of degree $n - 1$
$Q^*(s)$: Monic polynomial of degree $2n - 1$

Given $P(s)$ and $Q(s)$ that are coprime, and $Q^*(s)$, there exist $M(s)$ and $N(s)$ such that

$$Q(s)M(s) + P(s)N(s) = Q^*(s) \implies t_1(s), t_2(s) \text{ exist}$$
\[ R_p(s)(Z'_m(s) - t_1^*(s)) - (k_pZ_p(s))t_2^*(s) = R_m(s)\lambda(s)Z_p(s) \]
\[ Z'_m(s) = \det(sI - F_{n-1}) \]
\[
R_p(s)(Z'_m(s) - t_1^*(s)) - (k_pZ_p(s))t_2^*(s) = R_m(s)\lambda(s)Z_p(s) \\
Z'_m(s) = \text{det}(sI - F_{n-1})
\]
Analytic Part

\[ R_p(s)(Z'_m(s) - t_1^*(s)) - (k_pZ_p(s))t_2^*(s) = R_m(s)\lambda(s)Z_p(s) \]

\[ Z'_m(s) = \text{det}(sI - F_{n-1}) \]

\[
\begin{align*}
y_p & = \theta^*_2 \omega(t) \\
u(t) & = \theta^T(T)\omega(t)
\end{align*}
\]
\[
R_p(s)(Z'_m(s) - t_1^*(s)) - (k_pZ_p(s))t_2^*(s) = R_m(s)\lambda(s)Z_p(s)
\]

\[
Z'_m(s) = \det(sI - F_{n-1})
\]

\[
u(t) = \theta^T(t)\omega(t)
\]
\[
= \theta^*_T\omega(t) + \tilde{\theta}^T(t)\omega(t)
\]
Analytic Part

\[ R_p(s)(Z'_m(s) - t_1^*(s)) - (k_pZ_p(s))t_2^*(s) = R_m(s)\lambda(s)Z_p(s) \]
\[ Z'_m(s) = \det(sI - F_{n-1}) \]

\[
\begin{align*}
u(t) & = \theta^T(t)\omega(t) \\
& = \theta^*T\omega(t) + \tilde{\theta}^T(t)\omega(t) \\
\theta^* & = [k^*, \theta_1^*, \theta_{20}^*, \theta_2^*]^T, \quad \omega = [r, \omega_1^T, \omega_2^T]^T
\end{align*}
\]
Analytic Part

\[ R_p(s)(Z'_m(s) - t_1^*(s)) - (k_pZ_p(s))t_2^*(s) = R_m(s)\lambda(s)Z_p(s) \]
\[ Z'_m(s) = \text{det}(sI - F_{n-1}) \]

\[ \theta^* = [k^*, \theta_1^{*T}, \theta_20^{*T}, \theta_2^*]T, \quad \omega = [r, \omega_1^T, y_p, \omega_2^T]^T \]
Analytic Part

\[ R_p(s)(Z'_m(s) - t'_1(s)) - (k_pZ_p(s))t'_2(s) = R_m(s)\lambda(s)Z_p(s) \]
\[ Z'_m(s) = det(sI - F_{n-1}) \]

\[ u(t) = \theta^T(t)\omega(t) \]
\[ = \theta^*T\omega(t) + \tilde{\theta}^T(t)\omega(t) \]
\[ \theta^* = [k^*, \theta^*_1, \theta^*_{20}, \theta^*_2]^T, \omega = [r, \omega^*_1, y_p, \omega^*_2]^T \]
Error Model

\[ \theta^T \rightarrow k_p \rightarrow k_m z_m(s) R_m(s) \rightarrow e_1 \]

needs to be SPR

\[
\begin{align*}
\dot{k} &= -\text{sign}(k_p)e_1 r \\
\dot{\theta}_1 &= -\text{sign}(k_p)e_1 \omega_1 \\
\dot{\theta}_{20} &= -\text{sign}(k_p)e_1 y_p \\
\dot{\theta}_2 &= -\text{sign}(k_p)e_1 \omega_2
\end{align*}
\]

Problem: \( W_m(s) \) not SPR!
Several methods have been proposed in the literature. We will discuss one method -

( aanna@mit.edu)
Error Model

\[ \dot{k} = -\text{sign}(k_p)e_1 r \]
\[ \dot{\theta}_1 = -\text{sign}(k_p)e_1 \omega_1 \]
\[ \dot{\theta}_{20} = -\text{sign}(k_p)e_1 y_p \]
\[ \dot{\theta}_2 = -\text{sign}(k_p)e_1 \omega_2 \]

Problem: \( W_m(s) \) not SPR!
Several methods have been proposed in the literature. We will discuss one method - Augmented Error
Augmented Error: Swapping Lemma

Assume $k_p$ known.
Augmented Error: Swapping Lemma

Assume $k_p$ known. $e_2$: Auxiliary error

$$ e_2 = -W_m(s)[\theta^T \omega(t)] + \theta^T W_m(s)[\omega(t)] $$
Augmented Error: Swapping Lemma

Assume $k_p$ known. $e_2$: Auxiliary error

$$e_2 = -W_m(s)[\theta^T \omega(t)] + \theta^T W_m(s)[\omega(t)]$$
$$\epsilon_1 = e_1 + e_2 = \tilde{\theta}^T W_m(s)\omega = \tilde{\theta}^T \zeta$$

- Error Model 1

(aanna@mit.edu)
Augmented Error: Swapping Lemma

Assume $k_p$ known. $e_2$: Auxiliary error

$$e_2 = -W_m(s)[\theta^T \omega(t)] + \theta^T W_m(s)[\omega(t)]$$
$$= (-W_m(s)[\theta^* T \omega] + \theta^* T W_m(s)[\omega]) - W_m(s)[\tilde{\theta}^T \omega] + \tilde{\theta}^T W_m(s)[\omega]$$

$$\epsilon_1 = e_1 + e_2$$
$$= \tilde{\theta}^T W_m(s)\omega = \tilde{\theta}^T \zeta$$
Augmented Error: Swapping Lemma

Assume $k_p$ known. $e_2$: Auxiliary error

\[
e_2 = -W_m(s)[\theta^T \omega(t)] + \theta^T W_m(s)[\omega(t)]
= \left(-W_m(s)[\theta^T \omega] + \theta^T W_m(s)[\omega]\right) - W_m(s)[\tilde{\theta}^T \omega] + \tilde{\theta}^T W_m(s)[\omega]
\]

\[
\epsilon_1 = e_1 + e_2
= \tilde{\theta}^T W_m(s)\omega = \tilde{\theta}^T \zeta
\]
Augmented Error

Not easy to show that $\epsilon_1$ (and $e_1$) are bounded.

1. $\tilde{\theta} \in \mathcal{L}_\infty$. 
Augmented Error

$\tilde{\theta} \in L_{\infty}$. Converts NLTV system to an LTV system

Not easy to show that $\epsilon_1$ (and $e_1$) are bounded.
Not easy to show that $\epsilon_1$ (and $e_1$) are bounded.

1. $\tilde{\theta} \in \mathcal{L}_\infty$. Converts NLTV system to an LTV system
2. $\dot{\tilde{\theta}} \in \mathcal{L}_2$. 
Not easy to show that $\epsilon_1$ (and $e_1$) are bounded.

1. $\tilde{\theta} \in \mathcal{L}_\infty$. Converts NLTV system to an LTV system
2. $\dot{\tilde{\theta}} \in \mathcal{L}_2$. Converts LTV system to an almost time-invariant system
3. Proof by contradiction.
Not easy to show that $\epsilon_1$ (and $e_1$) are bounded.

1. $\tilde{\theta} \in \mathcal{L}_\infty$. Converts NLTV system to an LTV system
2. $\dot{\tilde{\theta}} \in \mathcal{L}_2$. Converts LTV system to an almost time-invariant system
3. Proof by contradiction. $\implies \omega \in \mathcal{L}_\infty$.
Proof of boundedness

\[ k(t) \]

\[ k(t) \]

\[ u \]

\[ \frac{k_p z_p}{R_p} \]

\[ \theta_1(t) \]

\[ F_{n-1}, g_{n-1} \]

\[ \theta_2(t) \]

\[ F_{n-1}, g_{n-1} \]

\[ \theta_{20}(t) \]

\[ y_p \]

\[ r \]

\[ r' \]

\[ y_m \]

\[ y_p \]

\[ e_1 \]

\[ e_2 \]

\[ \epsilon_1 \]

\[ \omega \]

\[ W_m(s) \]

\[ \tilde{\theta} \]

\[ \epsilon_1 \]

\[ \dot{\theta} = -\epsilon_1 \zeta \]

\[ V = \frac{1}{2} \tilde{\theta}^T \tilde{\theta} \]

\[ \dot{V} = -\epsilon_1 \tilde{\theta}^T \zeta = -\epsilon_1^2 \leq 0 \]
Proof of boundedness

\[
\dot{r} = k(t) \quad r' = \frac{k_{p}Z_{p}}{R_{p}}
\]

Plant

\[
\theta_{1}^{\top}(t) \quad F_{n-1}, g_{n-1}
\]

\[
\theta_{2}^{\top}(t) \quad F_{n-1}, g_{n-1}
\]

\[
\theta_{20}(t)
\]

\[
y_{p} = y_{m} + e_{1} + \epsilon_{1}
\]

\[
y_{p} = \omega W_{m}(s) \Rightarrow \tilde{\theta}^{\top} \in \mathcal{L}_{\infty}
\]

\[
\dot{\theta} = -\epsilon_{1} \zeta
\]

\[
V = \frac{1}{2} \tilde{\theta}^{\top} \tilde{\theta}
\]

\[
\dot{V} = -\epsilon_{1} \tilde{\theta}^{\top} \zeta = -\epsilon_{1}^{2} \leq 0 \quad \Rightarrow \quad \tilde{\theta} \in \mathcal{L}_{\infty}
\]
Proof of boundedness

\[ \dot{\theta} = -\epsilon_1 \zeta, \]
Proof of boundedness

\[
\dot{\theta} = -\epsilon_1 \zeta, \quad \implies \dot{\theta} \in \mathcal{L}_2
\]
Proof of boundedness

\[
\dot{\theta} = -\epsilon_1 \zeta, \quad \Rightarrow \quad \dot{\theta} \in \mathcal{L}_2
\]

Change to

\[
\dot{\theta} = -\frac{\epsilon_1 \zeta}{1 + \zeta^T \zeta + \omega^T \omega}
\]
Proof of boundedness

\[
\dot{\theta} = \frac{\epsilon_1 \zeta}{1 + \zeta^T \zeta + \omega^T \omega}
\]

\[
V = \frac{1}{2} \tilde{\theta}^T \tilde{\theta}
\]

\[
\dot{V} = -\epsilon_1 \frac{\tilde{\theta}^T \zeta}{1 + \zeta^T \zeta + \omega^T \omega}
\]
Proof of boundedness

\[ \dot{\theta} = -\frac{\epsilon_1 \zeta}{1 + \zeta^T \zeta + \omega^T \omega} \]
\[ V = \frac{1}{2} \tilde{\theta}^T \tilde{\theta} \]
\[ \dot{V} = -\epsilon_1 \frac{\tilde{\theta}^T \zeta}{1 + \zeta^T \zeta + \omega^T \omega} = -\epsilon_1 \frac{\epsilon_1^2}{1 + \zeta^T \zeta + \omega^T \omega} \leq \]
Proof of boundedness

\[ \dot{\theta} = -\frac{\epsilon_1 \zeta}{1 + \zeta^T \zeta + \omega^T \omega} \]

\[ V = \frac{1}{2} \tilde{\theta}^T \tilde{\theta} \]

\[ \dot{V} = -\epsilon_1 \frac{\tilde{\theta}^T \zeta}{1 + \zeta^T \zeta + \omega^T \omega} = -\frac{\epsilon_1^2}{1 + \zeta^T \zeta + \omega^T \omega} \leq 0 \]

\[ \Rightarrow \tilde{\theta} \in \mathcal{L}_\infty, \]
Proof of boundedness

\[
\begin{align*}
\dot{\theta} &= -\frac{\epsilon_1 \zeta}{1 + \zeta^T \zeta + \omega^T \omega} \\
V &= \frac{1}{2} \tilde{\theta}^T \tilde{\theta} \\
\dot{V} &= -\epsilon_1 \frac{\tilde{\theta}^T \zeta}{1 + \zeta^T \zeta + \omega^T \omega} = -\frac{\epsilon_1^2}{1 + \zeta^T \zeta + \omega^T \omega} \\
\Rightarrow \quad \tilde{\theta} &\in \mathcal{L}_\infty, \quad \hat{\theta} \in \mathcal{L}_2
\end{align*}
\]
Proof of boundedness

Proof by contradiction.

Let all signals grow in an unbounded manner.
Proof of boundedness

Proof by contradiction. Let all signals grow in an unbounded manner.