Problem Statement

Problem: \( \min_x f(x) \)

\( f(x) \) unknown

Identifying \( f(x) \) at all \( x \) is impossible
We convert \( f(x) \) into basis functions
Approximation

\[ f(x) = \sum_{i=1}^{N} a_i \sin(\omega_i x) \]

Fourier Series says that

\[ |f(x) - \sum_{i=1}^{N} a_i \sin(\omega_i x)| \leq \epsilon \]

Taylor Series says that

\[ |f(x) - \sum_{i=1}^{N} a_i x^i| \leq \epsilon, \quad \text{for } x \in [x_1, x_2] \]
Neural Networks

\[ \sum a_i \sin(\omega_i x), \sum a_i x^i : \text{linearly parameterized} \]

Departure: Neural Networks

Let \( \sigma (w_i^T x + \theta_i) = \sigma (w_i^T x) = \sigma_i (x); \Sigma = [\sigma_1(x), \sigma_2(x), \ldots, \sigma_N(x)]^T \)

\[ y = \alpha^T \Sigma(x) \]
Theorem

\( \sigma \) is any continuous sigmoidal function. Let

\[
\mathcal{N}(x) = \sum_{i=1}^{N} \alpha_i \sigma \left( w_i^T x \right).
\]

Then for any \( f \) that is continuous and \( \epsilon > 0 \), there exists \( \mathcal{N} \) such that

\[
\left| \mathcal{N}(x) - f(x) \right| < \epsilon, \quad \forall x \in X,
\]

(1)

where \( X \): compact set

Instead of (1), one can have

\[
\int_{X} \left| \mathcal{N}(x) - f(x) \right| dx < \epsilon
\]
Stone-Weierstrass Theorem

How do we adjust

- $w_i$ in $\Sigma(x)$?
- $\alpha_i$ in $\alpha^T$?

Use the back propagation algorithm:

\[
J = e^2 \\
\dot{\alpha} \propto -\frac{\partial J}{\partial \alpha} \\
\dot{w}_i \propto -\frac{\partial J}{\partial w_i}
\]
Back Propagation Algorithm

\[ e = \hat{y} - y = \sum_{i=1}^{N} \hat{\alpha}_i \sigma \left( \hat{w}_i^T x \right) - y \]

\[ \frac{\partial e}{\partial \alpha_i} \bigg|_{\hat{\alpha}_i} = \sigma \left( \hat{w}_i^T x \right); \quad \dot{\hat{\alpha}}_i = -\gamma e \frac{\partial e}{\partial \alpha_i} \bigg|_{\hat{\alpha}_i} \]

\[ \frac{\partial e}{\partial w_i} \bigg|_{\hat{w}_i} = \hat{\alpha}_i \sigma' \bigg|_{\hat{w}_i^T x} \cdot x; \quad \dot{\hat{w}}_i = -\gamma e \frac{\partial e}{\partial w_i} \bigg|_{\hat{w}_i} \]
Back Propagation Algorithm

\[ e = \hat{y} - y = \sum_{i=1}^{N} \hat{\alpha}_i \sigma (\hat{w}_i^T x) - y \]

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We did not discuss stability properties
Consider a very simple example:

\[ \dot{x}_p = -ax_p + f(x_p) + u \]
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\[ f(x_p) \approx \left( \sum_{j=1}^{N} \alpha_j \sigma(w_j x_p) \right) \]
Control of a Nonlinear Dynamic System

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Consider a very simple example:

\[ \dot{x}_p = -ax_p + \alpha \sigma(wx_p) + u \]

\[ u(t) = -\hat{\alpha} \sigma(\hat{w}x_p) + r \]

Define

\[ \epsilon = \sigma(wx_p) - \sigma(\hat{w}x_p) \quad \text{bounded} \]

Closed-loop Equations:

\[ \dot{x}_p = -ax_p + \alpha \sigma(wx_p) - \hat{\alpha} \sigma(\hat{w}x_p) + r \]
Control of a Simple Nonlinear Plant

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\dot{x}_p = -ax_p + \alpha \sigma(wx_p) + u
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Closed-loop Equations:

\[
\dot{x}_p = -ax_p + \alpha \sigma(w x_p) - \hat{\alpha} \sigma(\hat{w} x_p) + r \\
= -ax_p + (\alpha \sigma(w x_p) - \alpha \sigma(\hat{w} x_p) + \alpha \sigma(\hat{w} x_p) - \hat{\alpha} \sigma(\hat{w} x_p)) + r
\]
Control of a Simple Nonlinear Plant

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\dot{x}_p = -ax_p + \alpha \sigma(w x_p) + u \\
u(t) = -\hat{\alpha} \sigma(\hat{w} x_p) + r
\]

Define

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\epsilon = \sigma(w x_p) - \sigma(\hat{w} x_p) \quad \text{bounded}
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Closed-loop Equations:

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\dot{x}_p = -ax_p + \alpha \sigma(w x_p) - \hat{\alpha} \sigma(\hat{w} x_p) + r \\
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= -ax_p + \alpha \epsilon - \hat{\alpha} \sigma(\hat{w} x_p) + r
\]

Reference Model:

\[
\dot{x}_m = -ax_m + r
\]
Control of a Simple Nonlinear Plant

\[ \dot{x}_p = -a x_p + \alpha \sigma(w x_p) + u \]
\[ u(t) = -\hat{\alpha} \sigma(\hat{w} x_p) + r \]

Define

\[ \epsilon = \sigma(w x_p) - \sigma(\hat{w} x_p) \quad \text{bounded} \]

Closed-loop Equations:

\[ \dot{x}_p = -a x_p + \alpha \sigma(w x_p) - \hat{\alpha} \sigma(\hat{w} x_p) + r \]
\[ = -a x_p + (\alpha \sigma(w x_p) - \alpha \sigma(\hat{w} x_p) + \alpha \sigma(\hat{w} x_p) - \hat{\alpha} \sigma(\hat{w} x_p)) + r \]
\[ = -a x_p + \alpha \epsilon - \tilde{\alpha} \sigma(\hat{w} x_p) + r \]

Reference Model:

\[ \dot{x}_m = -a x_m + r \]

Error model:

\[ \dot{e} = -a e - \tilde{\alpha} \sigma(\hat{w} x_p) + \alpha \epsilon \]
Stability

Error model:

\[
\begin{align*}
\dot{e} &= -ae - \tilde{\alpha}\sigma(\hat{w}x_p) + \alpha\epsilon \\
\dot{\alpha} &= -e\sigma(\hat{w}x_p)
\end{align*}
\]
Stability

Error model:

\[
\begin{align*}
\dot{e} &= -ae - \tilde{\alpha} \sigma(\hat{w}x_p) + \alpha \epsilon \\
\dot{\tilde{\alpha}} &= -e \sigma(\hat{w}x_p)
\end{align*}
\]

\[
V = \frac{1}{2} (e^2 + \tilde{\alpha}^2)
\]

\[
\dot{V} \leq -ae^2 + \alpha e \epsilon
\]

Stability result:
Suppose \( r \) is persistently exciting with the property

\[
\frac{1}{T} \int_{t_1}^{t_1+T} |r(\tau)| d\tau \geq \frac{\alpha \epsilon}{a} + \delta
\]

then \( e(t) \) and \( \tilde{\alpha}(t) \) are bounded for all initial conditions \( e(t_0) \) and \( \tilde{\alpha}(t_0) \).
Control of a Nonlinear Dynamic System

\[ \dot{x}_p = -ax_p + f(x_p) + u \]
Control of a Nonlinear Dynamic System

\[ \dot{x}_p = -ax_p + f(x_p) + u \]

\[ f(x_p) \approx \left( \sum_{j=1}^{N} \alpha_j \sigma(w_j x_p) \right) \]
Control of a Nonlinear Dynamic System

\[ \dot{x}_p = -ax_p + f(x_p) + u \]

\[ f(x_p) \approx \left( \sum_{j=1}^{N} \alpha_j \sigma(w_j x_p) \right) \]

Consider the plant:

\[ \dot{x}_p = -ax_p + \left( \sum_{j=1}^{N} \alpha_j \sigma(w_j x_p) \right) + u \]

\[ u(t) = - \left( \sum_{j=1}^{N} \hat{\alpha}_j \sigma(\hat{w}_j x_p) \right) + r \]

\[ \epsilon = \left( \sum_{j=1}^{N} \sigma(w_j x_p) - \sigma(\hat{w}_j x_p) \right) \text{ bounded} \]
\[
\dot{x}_p = -ax_p + \left( \sum_{j=1}^{N} \alpha_j \sigma(w_j x_p) \right) + u
\]

\[
u(t) = -\left( \sum_{j=1}^{N} \hat{\alpha}_j \sigma(\hat{w}_j x_p) \right) + r
\]

\[
\epsilon = \left( \sum_{j=1}^{N} \sigma(w_j x_p) - \sigma(\hat{w}_j x_p) \right) \text{ bounded}
\]

\[
\dot{x}_p = -ax_p + \sum_{j=1}^{N} [\alpha_j \sigma(w_j x_p) - \hat{\alpha}_j \sigma(\hat{w}_j x_p)] + r
\]
\begin{align*}
\dot{x}_p &= -ax_p + \sum_{j=1}^{N} \left[ \alpha_j \sigma(w_j x_p) - \hat{\alpha}_j \sigma(\hat{w}_j x_p) \right] + r \\
&= -ax_p + \sum_{j=1}^{N} \alpha_j \epsilon - \hat{\alpha}_j \sigma(\hat{w}_j x_p) + r
\end{align*}

Error model:

\begin{align*}
\dot{e} &= -ae - \sum_{j=1}^{N} \alpha_j \epsilon - \hat{\alpha}_j \sigma(\hat{w}_j x_p) \\
\dot{\hat{\alpha}}_j &= e\sigma(\hat{w}_j x_p)
\end{align*}
Stability Analysis

Error model:

\[
\begin{align*}
\dot{e} &= -ae - \sum_{j=1}^{N} \alpha_j \epsilon - \tilde{\alpha}_j \sigma(\hat{w}_j x_p) \\
\dot{\tilde{\alpha}}_j &= e\sigma(\hat{w}_j x_p)) \\
V &= \frac{1}{2} \left( e^2 + \sum_{j=1}^{N} \tilde{\alpha}_j^2 \right) \\
\dot{V} &\leq -ae^2 + e \sum_{j=1}^{N} \alpha_j \epsilon
\end{align*}
\]
Stability Analysis

Error model:

\[ \dot{e} = -ae - \sum_{j=1}^{N} \alpha_j \epsilon - \tilde{\alpha}_j \sigma(\hat{w}_j x_p) \]

\[ \dot{\tilde{\alpha}}_j = e \sigma(\hat{w}_j x_p) \]

\[ V = \frac{1}{2} \left( e^2 + \sum_{j=1}^{N} \tilde{\alpha}_j^2 \right) \]

\[ \dot{V} \leq -ae^2 + e \sum_{j=1}^{N} \alpha_j \epsilon \]

\[ = -ae^2 + ed_0; \quad d_0 : \text{bounded} \]
Stability Result

Error model:

\[ \dot{e} = -ae - \sum_{j=1}^{N} \alpha_j \epsilon - \tilde{\alpha}_j \sigma(\hat{w}_j x_p) \]

\[ \dot{\tilde{\alpha}}_j = e \sigma(\hat{w}_j x_p) \]

Suppose \( r \) is persistently exciting with the property

\[ \frac{1}{T} \int_{t_1}^{t_1+T} |r(\tau)d\tau \geq \frac{d_0}{\alpha} + \delta \]

then \( e(t) \) and \( \tilde{\alpha}(t) \) are bounded for all initial conditions \( e(t_0) \) and \( \tilde{\alpha}(t_0) \).