2.153 Adaptive Control
Fall 2019
Lecture 6: States Accessible: Adaptive Control

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Error model approach

Relation between two main errors in adaptive systems:

\( \tilde{\theta} \): Parameter error,
\( e \): Tracking/Identification Error

The error model provides cues for determining the adaptive law.

Our goal with error models:
- Find an adaptive law for adjusting \( \tilde{\theta} \) that guarantees stability.
- depends on the relationship between \( \tilde{\theta} \) and \( e \).
- Learn how to prove stability using error models.
- Attempt to cast new adaptive identification and control problems as one of our error models.

We have seen two error models: Error Model 1 and Error Model 3.

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We have seen two error models: Error Model 1 and Error Model 3
Identification of a Vector Parameter - Error Model 1

Error Model 1: \( e = \tilde{\theta}^\top u \)

Adaptive law:
\[
\dot{\tilde{\theta}} = -eu
\]

Lyapunov function:
\[
V(\tilde{\theta}) = \frac{1}{2} \tilde{\theta}^\top \tilde{\theta}
\]
\[
\dot{V} = \tilde{\theta}^\top \dot{\tilde{\theta}} = \tilde{\theta}^\top \dot{\tilde{\theta}} = -e^2 \leq 0
\]

\( \Rightarrow \tilde{\theta}(t) \) is bounded for all \( t \geq t_0 \)
Error Model 3: \( \dot{e} = -ae + \tilde{\theta}^\top u \)

\[ u(t) \rightarrow \tilde{\theta}^\top \rightarrow \frac{1}{s+a} \rightarrow e(t) \]

\( \tilde{\theta} \): parameter error

Adaptive law:

\[ \dot{\tilde{\theta}} = -eu \]

Lyapunov function:

\[ V(e, \tilde{\theta}) = \frac{1}{2} \left( e^2 + \tilde{\theta}^\top \tilde{\theta} \right) \]

\[ \dot{V} = e \dot{e} + \tilde{\theta}^\top \dot{\tilde{\theta}} \]

\[ = -ae^2 \leq 0 \]

\( \Rightarrow e(t) \) and \( \tilde{\theta}(t) \) are bounded for all \( t \geq t_0 \)
Adaptive Control of a First-Order Plant

Leads to Error model 3:

\[ \dot{e} = a_m e + k_p \tilde{\theta}^\top(t) \omega \]

\[ \omega = \begin{bmatrix} x_p \\ r \end{bmatrix}, \quad \tilde{\theta} = \begin{bmatrix} \tilde{\theta}(t) \\ \tilde{k}(t) \end{bmatrix} \]
Certainty Equivalence Principle - Step 2

Error model:

\[ \dot{e} = a_m e + k_p \bar{\theta}^T (t) \omega \]

(slightly modified) Lyapunov function:

\[ V = \frac{1}{2} \left( e^2 + |k_p| \bar{\theta}^T \bar{\theta} \right) \]

Leads to

\[ \dot{V} = e \dot{e} + |k_p| \bar{\theta}^T \dot{\bar{\theta}} \]

Adaptive law:

\[ \dot{\bar{\theta}} = - \text{sign}(k_p) e \omega \]

\[ \dot{V} = a_m e^2 \leq 0 \]

\[ \Rightarrow e_t \text{ and } \bar{\theta}_t \text{ are bounded for all } t \geq t_0 \]
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Error model: \[ \dot{e} = a_m e + k_p \tilde{\theta}^T (t) \omega \]

(slightly modified) Lyapunov function: \[ V = \frac{1}{2} \left( e^2 + |k_p| \tilde{\theta}^T \tilde{\theta} \right) \]

Leads to

\[ \dot{V} = e \dot{e} + |k_p| \tilde{\theta}^T \tilde{\theta} \]

\[ = a_m e^2 + \tilde{\theta}^T (k_p \omega + |k_p| \dot{\tilde{\theta}}) \]
Certainty Equivalence Principle - Step 2

Error model:  \[ \dot{e} = a_m e + k_p \tilde{\theta}^T(t) \omega \]

(slightly modified) Lyapunov function:  \[ V = \frac{1}{2} \left( e^2 + |k_p| \tilde{\theta}^T \tilde{\theta} \right) \]

Leads to

\[ \dot{V} = e \dot{e} + |k_p| \tilde{\theta}^T \dot{\tilde{\theta}} \]

\[ = a_m e^2 + \tilde{\theta}^T (k_p e \omega + |k_p| \dot{\tilde{\theta}}) \]

Adaptive law:  \[ \dot{\tilde{\theta}} = -\text{sign}(k_p) e \omega \]

\[ \dot{V} = a_m e^2 \leq 0 \]

\[ \Rightarrow e(t) \text{ and } \tilde{\theta}(t) \text{ are bounded for all } t \geq t_0 \]
Signal Norms

\( \mathcal{L}_p \) Norm

\[ \|x(t)\|_{L_p} = \left( \int_0^t \|x(\tau)\|^p d\tau \right)^{\frac{1}{p}} \]

\( \mathcal{L}_1 \) Norm

\[ \|x(t)\|_{L_1} = \int_0^t \|x(\tau)\| d\tau \]

\( \mathcal{L}_2 \) Norm

\[ \|x(t)\|_{L_2} = \sqrt{\int_0^t \|x(\tau)\|^2 d\tau} \]

\( \mathcal{L}_\infty \) Norm

\[ \|x(t)\|_{L_\infty} = \sup_t \|x(t)\| \]

\( V > 0, \quad \dot{V} = a_m e^2 \leq 0 \Rightarrow (i) e \in \mathcal{L}_\infty, \quad \tilde{\theta} \in \mathcal{L}_\infty, \quad (ii) e \in \mathcal{L}^2 \)
Adaptive Control of a First-Order Plant

Convergence of $e$ to zero:

- $e \in \mathcal{L}^\infty$ and $\tilde{\theta} \in \mathcal{L}^\infty$
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- $e \in \mathcal{L}^\infty$ and $\tilde{\theta} \in \mathcal{L}^\infty$
- For all bounded inputs $r$, $x_m$ is bounded
- $x_p = x_m + e$
Adaptive Control of a First-Order Plant

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- $\dot{V} = a_m e^2 \leq 0 \Rightarrow e \in \mathcal{L}^2$
- $\dot{e} = a_m e + k_p \tilde{\theta}^\top (t) \omega \Rightarrow \dot{e} \in \mathcal{L}^\infty$
Barbalat’s Lemma

Lemma 2.12 (Barbalat’s) page 85

(i) If \( f : \mathbb{R}^+ \rightarrow \mathbb{R} \) is uniformly continuous for \( t \geq 0 \)
(ii) And if \( \lim_{t \to \infty} \int_0^t |f(\tau)|d\tau \) exists and is finite

Then \( \lim_{t \to \infty} f(t) = 0 \)
Barbalat’s Lemma

**Lemma 2.12 (Barbalat’s) page 85**

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**Corollary** If $g \in \mathcal{L}^2 \cap \mathcal{L}^\infty$, and $\dot{g}$ is bounded, then $\lim_{t \to \infty} g(t) = 0$. 
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Corollary If $g \in L^2 \cap L^\infty$, and $\dot{g}$ is bounded, then $\lim_{t \to \infty} g(t) = 0$.

- $e \in L^\infty$ and $\tilde{\theta} \in L^\infty$
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Corollary If $g \in \mathcal{L}^2 \cap \mathcal{L}^\infty$, and $\dot{g}$ is bounded, then $\lim_{t \to \infty} g(t) = 0$.

- $e \in \mathcal{L}^\infty$ and $\tilde{\theta} \in \mathcal{L}^\infty$
- $\dot{V} = a_m e^2 \leq 0 \Rightarrow e \in \mathcal{L}^2$
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Corollary If $g \in \mathcal{L}^2 \cap \mathcal{L}^\infty$, and $\dot{g}$ is bounded, then $\lim_{t \to \infty} g(t) = 0$.

- $e \in \mathcal{L}^\infty$ and $\tilde{\theta} \in \mathcal{L}^\infty$
- $\dot{V} = a_m e^2 \leq 0 \Rightarrow e \in \mathcal{L}^2$
- $\dot{e} = a_m e + k_p \tilde{\theta}^T (t) \omega \Rightarrow \dot{e} \in \mathcal{L}^\infty$
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Corollary If \( g \in \mathcal{L}^2 \cap \mathcal{L}^\infty \), and \( \dot{g} \) is bounded, then \( \lim_{t \to \infty} g(t) = 0 \).

- \( e \in \mathcal{L}^\infty \) and \( \tilde{\theta} \in \mathcal{L}^\infty \)
- \( \dot{V} = a_m e^2 \leq 0 \Rightarrow e \in \mathcal{L}^2 \)
- \( \dot{e} = a_m e + k_p \tilde{\theta}^\top(t)\omega \Rightarrow \dot{e} \in \mathcal{L}^\infty \)
- Barbalat’s lemma \( \Rightarrow \lim_{t \to \infty} e(t) = 0 \)
Adaptive Control of Higher Order Plants (with a single input)
Example

\[ m\ddot{x} + b\dot{x} + kx = u \]

\( m, b, k \) unknown. Find \( u \) so that (i) \( x(t) \to 0 \), or (ii) \( x(t) \to x_m \)
Example

\[ m\ddot{x} + b\dot{x} + kx = u \]

\(m, b, k\) unknown. Find \(u\) so that (i) \(x(t) \to 0\), or (ii) \(x(t) \to x_m\)

\[ X_p = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \quad \dot{X}_p = A_p X_p + b_p u \]

\[ A_p = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \quad b_p = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \]
# States Accessible - Stabilization

**Plant:** \[
\dot{X}_p = A_p X_p + b_p u
\]

**Reference Model:** \[
\ddot{x}_m + 2\zeta\omega_n \dot{x}_m + \omega_n^2 x_m = \omega_n^2 r
\]

\[
X_m = \begin{bmatrix} x_m \\ \dot{x}_m \end{bmatrix}, \quad A_m = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \quad b_m = \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix}
\]

\[
e = X_p - X_m
\]

Choose \( u \) so that \( e(t) \to 0 \) as \( t \to \infty \). \( b_p, A_p \) are unknown.
Certainty Equivalence Principle - $b_p$ known

Step 1: **Algebraic Part**: Find a solution to the problem when parameters are known.

Plant:  \[ \dot{X}_p = A_p X_p + b_p u \]
Certainty Equivalence Principle - $b_p$ known

Step 1: **Algebraic Part:** Find a solution to the problem when parameters are known.

Plant: \[ \dot{X}_p = A_p X_p + b_p u \]

Controller: \[ u = \theta_T^c X_p + r \]
Certainty Equivalence Principle - $b_p$ known

Step 1: **Algebraic Part:** Find a solution to the problem when parameters are known.

\[
\begin{align*}
\text{Plant:} & \quad \dot{X}_p = A_p X_p + b_p u \\
\text{Controller:} & \quad u = \theta_c^T X_p + r \\
\text{Closed-loop:} & \quad \dot{X}_p = [A_p + b_p \theta_c^T] X_p + b_p r
\end{align*}
\]
Certainty Equivalence Principle - $b_p$ known

Step 1: **Algebraic Part**: Find a solution to the problem when parameters are known.

- **Plant**: $\dot{X}_p = A_p X_p + b_p u$
- **Controller**: $u = \theta^T_c X_p + r$
- **Closed-loop**: $\dot{X}_p = [A_p + b_p \theta^T_c] X_p + b_p r$

**Matching conditions**: $A_p + b_p \theta^* T = A_m$
Certainty Equivalence Principle - $b_p$ known

Step 1: **Algebraic Part:** Find a solution to the problem when parameters are known.

Plant: \[
\dot{X}_p = A_p X_p + b_p u
\]

Controller: \[
u = \theta^T_c X_p + r
\]

Closed-loop: \[
\dot{X}_p = \left[A_p + b_p \theta^*_T\right] X_p + b_p r
\]

Matching conditions: \[
A_p + b_p \theta^*_T = A_m
\]

Solution: \[
\theta_c = \theta^*
\]
Certainty Equivalence Principle - \( b_p \) known

Step 1: **Algebraic Part:** Find a solution to the problem when parameters are known.

Plant: \[
\dot{X}_p = A_p X_p + b_p u
\]

Controller: \[
u = \theta^T_c X_p + r\]

Closed-loop: \[
\dot{X}_p = \left[ A_p + b_p \theta^T_c \right] X_p + b_p r
\]

Matching conditions: \[
A_p + b_p \theta^*T = A_m
\]

Solution: \[
\theta_c = \theta^*
\]

Step 2: **Analytic Part:**

Controller: \[
u = \theta^T(t) X_p + r\]

Closed-loop: \[
\dot{X}_p = \left[ A_p + b_p \theta^T(t) \right] X_p + b_p r
\]

\[
= A_m X_p + b_p \tilde{\theta}^T X_p + b_p r
\]
Lyapunov functions and Linear Time-invariant Systems
(see section 2.4.4)

LTI System: \[ \dot{x} = A_m x \]
Lyapunov functions and Linear Time-invariant Systems (see section 2.4.4)

LTI System: \[
\dot{x} = A_m x \\
V = (x^T P x)
\]
Lyapunov functions and Linear Time-invariant Systems (see section 2.4.4)

LTI System: \[ \dot{x} = A_m x \]

\[ V = (x^T P x) \]

\[ \dot{V} = x^T [A_m^T P + P A_m] x \]
Lyapunov functions and Linear Time-invariant Systems  
(see section 2.4.4)

LTI System:  \[ \dot{x} = A_m x \]

\[
V = (x^T P x) \\
\dot{V} = x^T [A_m^T P + P A_m] x \\
\leq -x^T Q x \leq 0
\]
Error Model 2

Error equation: \[ \dot{e} = A_m e + b_p \tilde{\theta}^T X_p \]
Error equation: \[ \dot{e} = A_m e + b_p \tilde{\theta}^T X_p \]

\[ V = \left( e^T P e + \tilde{\theta}^T \tilde{\theta} \right) \]

\[ \dot{V} = e^T [A_m^T P + PA_m] e + 2e^T P b_p \tilde{\theta}^T X_p + 2\tilde{\theta}^T \tilde{\theta} \]
Error Model 2

Error equation: \[ \dot{e} = A_m e + b_p \tilde{\theta}^T X_p \]

\[ V = \left( e^T P e + \tilde{\theta}^T \tilde{\theta} \right) \]

\[ \dot{V} = e^T [A_m^T P + P A_m] e + 2e^T P b_p \tilde{\theta}^T X_p + 2\tilde{\theta}^T \dot{\tilde{\theta}} \]

\[ = -e^T Q e \quad \text{if } \dot{\tilde{\theta}} = -e^T P b_p X_p \]
Error Model 2

Error equation: \[ \dot{e} = A_m e + b_p \tilde{\theta}^T X_p \]

\[ V = \left( e^T P e + \tilde{\theta}^T \tilde{\theta} \right) \]

\[ \dot{V} = e^T [A_m^T P + PA_m] e + 2e^T P b_p \tilde{\theta}^T X_p + 2\tilde{\theta}^T \dot{\tilde{\theta}} \]

\[ = -e^T Q e \quad \text{if } \dot{\tilde{\theta}} = -e^T P b_p X_p \]

\[ \leq 0 \]

\[ \Rightarrow e(t) \quad \text{and} \quad \tilde{\theta}(t) \text{ are bounded for all } t \geq t_0 \]
Error Model 2

Error equation: \[ \dot{e} = A_m e + b_p \tilde{\theta}^T X_p \]

\[ V = \left( e^T Pe + \tilde{\theta}^T \tilde{\theta} \right) \]

\[ \dot{V} = e^T [A_m^T P + PA_m] e + 2e^T P b_p \tilde{\theta}^T X_p + 2\tilde{\theta}^T \dot{\tilde{\theta}} \]

\[ \leq 0 \]

\[ \Rightarrow e(t) \quad \text{and} \quad \tilde{\theta}(t) \quad \text{are bounded for all} \quad t \geq t_0 \]

\[ \lim_{t \to \infty} e(t) = 0 \quad \text{from Barbalat's Lemma} \]
Certainty Equivalence Principle - $b_p$ unknown

Step 1: Algebraic Part:

Plant: \[ \dot{X}_p = A_p X_p + b_p u \]

Controller: \[ u = \theta^T c X_p + k_c r \]

Closed-loop: \[ \dot{X}_p = [A_p + b_p \theta^T c] X_p + b_p k_c r \]
Certainty Equivalence Principle - $b_p$ unknown

Step 1: Algebraic Part:

- **Plant:** $\dot{X}_p = A_p X_p + b_p u$
- **Controller:** $u = \theta_c^T X_p + k_c r$
- **Closed-loop:** $\dot{X}_p = \left[ A_p + b_p \theta_c^T \right] X_p + b_p k_c r$

Matching conditions:

$$A_p + b_p \theta^* T = A_m; \quad b_p k^* = b_m$$
Certainty Equivalence Principle - \( b_p \) unknown

Step 1: Algebraic Part:

Plant: \[ \dot{X}_p = A_p X_p + b_p u \]

Controller: \[ u = \theta_c^T X_p + k_c r \]

Closed-loop: \[ \dot{X}_p = \left[ A_p + b_p \theta_c^{*T} \right] X_p + b_p k_c r \]

Matching conditions: \[ A_p + b_p \theta^{*T} = A_m; \; b_p k^{*} = b_m \]

Solution: \[ \theta_c = \theta^{*}, \; k_c = k^{*} \]
Certainty Equivalence Principle - $b_p$ unknown

Step 1: Algebraic Part:

Plant: $\dot{X}_p = A_p X_p + b_p u$

Controller: $u = \theta^T_c X_p + k_c r$

Closed-loop: $\dot{X}_p = \left[ A_p + b_p \theta^T_c \right] X_p + b_p k_c r$

Matching conditions: $A_p + b_p \theta^* T = A_m; \ b_p k^* = b_m$

Solution: $\theta_c = \theta^*, \ k_c = k^*$

Step 2: Analytic Part:

Controller: $u = \theta^T(t) X_p + k(t) r$
Certainty Equivalence Principle - $b_p$ unknown

Step 1: Algebraic Part:

Plant: \[ \dot{X}_p = A_p X_p + b_p u \]
Controller: \[ u = \theta_c^T X_p + k_c r \]
Closed-loop: \[ \dot{X}_p = \left[ A_p + b_p \theta_c^T \right] X_p + b_p k_c r \]

Matching conditions: \[ A_p + b_p \theta^*^T = A_m; \ b_p k^* = b_m \]
Solution: \[ \theta_c = \theta^*, \ k_c = k^* \]

Step 2: Analytic Part:

Controller: \[ u = \theta^T(t) X_p + k(t) r \]
Closed-loop: \[ \dot{X}_p = \left[ A_p + b_p \theta^T(t) \right] X_p + b_p (k^* + \tilde{k}) r \]
Certainty Equivalence Principle - $b_p$ unknown

Step 1: Algebraic Part:

- **Plant:** $\dot{X}_p = A_p X_p + b_p u$
- **Controller:** $u = \theta_c^T X_p + k_c r$
- **Closed-loop:** $\dot{X}_p = [A_p + b_p \theta_c^T] X_p + b_p k_c r$

**Matching conditions:**

$A_p + b_p \theta^* T = A_m; \ b_p k^* = b_m$

**Solution:**

$\theta_c = \theta^*, \ k_c = k^*$

Step 2: Analytic Part:

- **Controller:** $u = \theta^T(t) X_p + k(t) r$
- **Closed-loop:** $\dot{X}_p = [A_p + b_p \theta^T(t)] X_p + b_p (k^* + \tilde{k}) r$

$$= A_m X_p + b_p \left( \tilde{\theta}^T X_p + \tilde{k} r \right) + b_m r$$

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Error Model 2

Error equation:  
\[ \dot{e} = A_m e + b_p \left( \tilde{\theta}^T X_p + \tilde{k} r \right) \]
Error Model 2

Error equation: $\dot{e} = A_m e + b_p \left( \tilde{\theta}^T X_p + \tilde{k} r \right)$

$V = e^T P e + \tilde{\theta}^T \tilde{\theta} + \tilde{k}^2$

$\dot{V} = e^T [A_m^T P + P A_m] e + 2e^T P b_p \tilde{\theta}^T X_p + 2\tilde{\theta}^T \dot{\tilde{\theta}}$

$+ 2e^T P b_p \tilde{k} r + 2\tilde{k} \dot{\tilde{k}}$

$= -e^T Q e \quad \text{if } \dot{\tilde{\theta}} = -e^T P b_p X_p, \quad \dot{\tilde{k}} = -e^T P b_p r$

But $b_p$ is unknown.
Error Model 2

Error equation: 
\[ \dot{e} = A_m e + b_p \left( \tilde{\theta}^T X_p + \tilde{k} r \right) \]

\[ V = e^T P e + \tilde{\theta}^T \tilde{\theta} + \tilde{k}^2 \]

\[ \dot{V} = e^T [A_m^T P + P A_m] e + 2 e^T P b_p \tilde{\theta}^T X_p + 2 \tilde{\theta}^T \dot{\tilde{\theta}} + 2 e^T P b_p \tilde{k} r + 2 \tilde{k} \dot{\tilde{k}} \]

\[ = -e^T Q e \quad \text{if} \quad \dot{\tilde{\theta}} = -e^T P b_p X_p, \quad \dot{\tilde{k}} = -e^T P b_p r \]

But \( b_p \) is unknown. 
\[ b_p = k^* b_m \]
Error Model 2

Error equation: \[ \dot{e} = A_m e + b_p (\hat{\theta}^T X_p + \tilde{k} r) \]

\[ V = e^T P e + \hat{\theta}^T \hat{\theta} + \tilde{k}^2 \]

\[ \dot{V} = e^T [A_m^T P + P A_m] e + 2 e^T P b_p \hat{\theta}^T X_p + 2 \hat{\theta}^T \hat{\dot{\theta}} + 2 e^T P b_p \tilde{k} r + 2 \tilde{k} \dot{\tilde{k}} \]

\[ = -e^T Q e \quad \text{if} \quad \dot{\hat{\theta}} = -e^T P b_p X_p, \quad \dot{\tilde{k}} = -e^T P b_p r \]

But \( b_p \) is unknown.

Choose \[ \dot{\hat{\theta}} = -\text{sign}(k^*) e^T P b_m X_p, \quad \dot{\tilde{k}} = -\text{sign}(k^*) e^T P b_m r \]

\[ \Rightarrow V = \frac{1}{2} \left( e^T P e + |k^*| \left( \hat{\theta}^T \hat{\theta} + \tilde{k}^2 \right) \right), \quad \dot{V} = -e^T Q e \leq 0 \]

\[ \Rightarrow e(t), \hat{\theta}(t), \quad \text{and} \quad \tilde{k}(t) \quad \text{are bounded for all} \quad t \geq t_0 \]

From Barbalat’s lemma,
Overall Adaptive System

\begin{align*}
    b_p k^* &= b_m \\
    A_p + b_p \theta^{\ast\top} &= A_m \\
    \dot{\theta} &= -\text{sign}(k^*) e^\top P b_m X_p \\
    \dot{k} &= -\text{sign}(k^*) e^\top P b_m r
\end{align*}