Improved Transient Response in Adaptive Control
Using Projection Algorithms and Closed Loop
Reference Models

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This paper explores the properties of adaptive systems with closed–loop reference models. Historically, reference models in adaptive systems run open–loop in parallel with the plant and controller, using no information from the plant or controller to alter the trajectory of the reference system. Closed–loop reference models on the other hand use information from the plant to alter the reference trajectory. Using the extra design freedom available in closed–loop reference models, we design adaptive controllers that are (a) stable, and (b) have improved transient properties. Numerical studies that complement theoretical derivations are also reported.

I. Introduction

The central element of any adaptive systems is online parameter adjustment. This is usually accomplished by having a plant, determined by a dynamic model, along with a controller with adaptive parameters designed to compensate for the plant’s actions, follow a reference model. The resulting error between the reference model and the plant is used to adjust the adaptive parameter.

Open–loop reference models have been the backbone of adaptive control for the past four decades [1, 2] where modifications to the adaptive control law were first added for stability in the presence of bounded disturbances [3–5] and semi–global stability in the presence of unmodeled dynamics [6, 7]. Analysis and design techniques have also been introduced for stability in the presence of time varying systems [8–11].

In this paper, in contrast to open-loop reference models, a closed–loop reference model structure is proposed [12]. Taking cues from adaptive identifiers [13] and linear Luenberger observers [14], a feedback of the tracking error is introduced in the reference model structure as well. The associated feedback gain introduces an additional degree of freedom and is utilized in improving the adaptive system performance.

The most important property of the closed-loop reference model, apart from retaining the stability of the overall adaptive system, is its ability to guarantee an improved transient performance. Several attempts have been made over the past few decades to design adaptive systems with improved transient performance. Investigations related to combined/composite direct and indirect adaptation showed promise as simulations portrayed smoother transients when compared to either direct or indirect learning alone [15–17]. While the results of these papers established stability of these combined schemes, no rigorous guarantees of improved transient performance were provided, and remains a conjecture [18].

More recently closed-loop reference models have been suggested in [19–23]. It is well known that in plants where states are accessible, such reference models can be introduced without compromising stability of the adaptive system. Here too, we show that stability can be established with closed–loop reference models.

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The overall adaptive system with the same reference model is shown to be robust to disturbances and time-varying parameters, with the tracking error converging exponentially to a compact set. Both in the presence and absence of these perturbations, it is shown that there is significant flexibility in the design with the closed-loop feature in the reference model, which can be utilized for controlling the transients. We show in particular that a “water–bed” effect can result for poor choices of the reference model parameters and propose an optimal parameter selection that ensures satisfactory transients, and represents an improvement over the discussions in [22].

II. Notation

All norms unless otherwise noted are the 2–norm and the induced 2–norm [24, 25]. The variable \( t \in \mathbb{R}_+ \) denotes time throughout and for a differentiable function \( x \), \( \dot{x} \) is equivalent to \( \dot{x} \). Parameters explicit time dependence \((t)\) is used upon introduction and then omitted thereafter except for emphasis. Big–O notation \( O(\cdot) \) is defined as follows, for \( f(x), g(x) \in \mathbb{R} \rightarrow \mathbb{R} \) \( f(x) = O(g(x)) \) if \( \exists M > 0 \) such that \( \|f(x)\| \leq M\|g(x)\| \) for all \( x \in \mathcal{X} \subset \mathbb{R} \).

III. Full States Accessible

A. Basic Adaptive Structure

Consider the dynamics
\[
\dot{x}(t) = A_p x(t) + bu(t)
\] (1)
where \( x(t) \in \mathbb{R}^n \) is the state vector, \( u(t) \in \mathbb{R} \) is the input, \( A_p \in \mathbb{R}^{n \times n} \) is unknown and \( b \in \mathbb{R}^n \) is known. Our goal is to design the control input such that \( x(t) \) follow the reference model state \( x_m(t) \in \mathbb{R}^n \) defined by the following dynamics
\[
\dot{x}_m(t) = A_m x_m(t) + b r(t) - L(x(t) - x_m(t))
\] (2)
where \( A_m \in \mathbb{R}^{n \times n} \) is Hurwtitz, and is such that a parameter vector, \( \theta^* \), exists that solves
\[
A_m = A_p + b \theta^T,
\] (3)
and \( r(t) \in \mathbb{R} \) is a reference command. We note that compared to the classical adaptive control case, we have introduced an additional feedback term \( L(x - x_m) \) in the reference model. We refer to \( L \in \mathbb{R}^{n \times n} \) as the Luenberger–gain. Note that when \( L = 0 \), the classical open–loop reference model is recovered.

The control input is chosen in a standard form to be
\[
u = \theta^T(t)x + r
\] (4)
where \( \theta(t) \in \mathbb{R}^n \) is the adaptive control gain with the update law
\[
\dot{\theta} = -\Gamma xe^T P b
\] (5)
with \( \Gamma = \text{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n) > 0, e = x - x_m \) the model following error and \( P = P^T > 0 \) as the solution to
\[
\tilde{A}_m^T P + P \tilde{A}_m = -I_{n \times n},
\] (6)
\[
\tilde{A}_m \triangleq A_m + L.
\] (7)
The underlying error model in this case is given by
\[
\dot{e} = \tilde{A}_m e + \tilde{b}\dot{\theta}(t) x
\] (8)
where \( \tilde{\theta}(t) = \theta(t) - \theta^* \) is the parameter error.

**Theorem 1.** The closed-loop adaptive system with (1), (2), (4) and (5) is globally stable with \( e(t) \) tending to zero asymptotically, under the matching conditions in (3).

**Proof.** It is straightforward to show using (5) and (8) that
\[
V(e, \theta) = e^T P e + \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}
\] (9)
is a Lyapunov function. Barbalat’s lemma ensures convergence of \( e(t) \) to zero. \( \square \)
B. Projection Algorithm

Before we evaluate the benefits of closed-loop reference models, we introduce a modification in the adaptive law to ensure robustness properties. For this purpose, we assume that a known \( \theta^* \) exists such that
\[
\| \theta^* \| \leq \theta^*_{\text{max}}.
\]

The projection based adaptive law, which replaces (5), is given by
\[
\dot{\theta}(t) = \text{Proj}_\Gamma(\theta(t), -xe^TPb, f)
\]  
(10)

where the \( \Gamma \)–projection function is defined as in Appendix A and
\( f \) is a convex function given by
\[
f(\theta; \vartheta, \varepsilon) = \| \theta \|^2 - \vartheta^2 \frac{2}{\varepsilon \theta^2 - \varepsilon^2}
\]  
(11)

with \( \vartheta \) chosen such that
\( \vartheta = \theta^*_{\text{max}} \) and \( \varepsilon > 0 \).

**Definition 1.** Given a Hurwitz matrix \( A_m \in \mathbb{R}^{n \times n} \)
\[
\sigma = \min_i \text{real}(\lambda_i(A_m))
\]  
\newline
\[
a = \| A_m \|
\]  
\newline
\[
s = -\min_i \left( \lambda_i \left( A_m + A^T_m \right) \right) / 2
\]  
(12)

For ease of exposition, throughout the paper, we choose \( L \) in (2) and \( \Gamma \) in (10) and (42) as follows:
\[
L \triangleq -\ell I_{n \times n}
\]  
(13)
\[
\Gamma \triangleq \gamma I_{n \times n}
\]  
(14)

**Lemma 2.** With \( L \) and \( \Gamma \) chosen as in (13) and (14), \( A_m \) Hurwitz with constants \( \sigma \) and \( a \) as defined in (12), then \( P \) from (6) can be upper bounded as
\[
\| P \| \leq \frac{m^2}{\sigma + 2\ell}
\]  
\newline
\[
\min_i \lambda_i(P) \geq \frac{1}{2(s + \ell)}
\]  
(15)

where \( m = (1 + 4\kappa)^{n-1} \) and \( \kappa \triangleq \frac{a}{\sigma} \). The proof is located in Appendix B.

**Theorem 3.** Given the uncertain system of equation in (1) with the reference model in (2), the controller in (4), the adaptive tuning law in (10)-(11) and (42), \( \theta(0) \) chosen such that \( \| \theta(0) \| \leq \vartheta + \varepsilon \), and with choices as in (13)-(14) the Lyapunov candidate in (9) converges exponentially to a set \( E \) given by
\[
E \triangleq \{ (e, \tilde{\theta}) \| e \|^2 \leq \beta_1 \tilde{\theta}_{\text{max}}^2, \| \tilde{\theta} \| \leq \tilde{\theta}_{\text{max}} \}
\]  
(16)

where \( \tilde{\theta}_{\text{max}} = 2\vartheta + \varepsilon \) and \( \beta_1 \) is given in (50) in Appendix C with an order of magnitude
\[
\beta_1 = O \left( \frac{\ell}{\gamma} \right)
\]  
(17)

with the exponent
\[
\alpha_1 \triangleq \frac{\sigma + 2\ell}{m^2}
\]  
(18)

**Proof.** see Appendix C.

C. Transients and Performance

We now show that the overall adaptive system has a transient performance that can be suitably shaped using the free parameters \( \ell \) and \( \gamma \) of the adaptive system. We restrict our attention to a specific measure of transient performance in this paper, which corresponds to \( \dot{u}(t) \). In particular, we will derive upper bounds
Theorem 3, can be shown to be

\[ \tau_1(l) = \frac{1}{\sigma + \ell} \quad \text{and} \quad \tau_2 = \frac{1}{\sigma + \ell}. \] (19)

We note that \( \tau_1 \) and \( \tau_2 \) are the time constants for the error dynamics in (8) and reference model in (2), respectively. We also define a third time–constant \( \tau_1' \) associated with the Lyapunov function, which from Theorem 3, can be shown to be

\[ \tau_1'(l) = \frac{2m^2}{\sigma + 2\ell}. \] (20)

from (18). We also define two constants

\[ \epsilon_1(l) = \frac{\|b\|\tilde{\theta}_{\max}}{\sigma + \ell + a_\theta}, \]

\[ \epsilon_2(\ell) = \exp(a_\theta 4\tau_1(\ell)) - 1 \] (21)

where \( a_\theta = \|A_m + b\tilde{\theta}\| \).

We define \( \ell^* \) as a constant that satisfies the inequalities

\[ \tau_1(\ell^*) < \frac{1}{10} \tau_2 \quad \text{and} \quad \tau_1'(\ell^*) < \frac{1}{10} \tau_2 \] (22)

and

\[ \epsilon_1(\ell^*) < \epsilon \quad \text{and} \quad \epsilon_2(\ell^*) < \epsilon \] (23)

where \( \epsilon \) is an arbitrarily small constant. It is easy to show that such an \( \ell^* \) always exists.

**Assumption 1.** The reference signal is assumed to be piece–wise constant and the reference model state vector is initialized at \( x_m(0) = 0 \).

**Theorem 4.** Let Assumptions 1 hold, and \( \ell = \ell^* \). Given arbitrary initial conditions in \( x(0) \in \mathbb{R}^n \) and \( \|\theta(0)\| \leq \theta + \varepsilon \), the derivative satisfies the following four inequalities:

\[ |\dot{u}(0)| \leq O\left(\frac{\gamma}{\sigma + \ell}\right) \|e(0)\|^3 + O(1)(\|\theta(0)\|^2 + \|\theta(0)\|)\|e(0)\| \] (24)

\[ |\dot{u}(0^+)| \leq O\left(\frac{\gamma}{\sigma + \ell}\right) \|e(0)\|^3 + O(1)(\|\tilde{\theta}_{\max}\|^2 + \|\tilde{\theta}_{\max}\|)\|e(0)\| \] (25)

\[ |\dot{u}(4\tau_1)| \leq O\left(\epsilon_{\tau_1}\frac{\gamma}{\sigma + \ell}\right) \|e(0)\|^3 + O(1 + \epsilon_1)(\tilde{\theta}_{\max}^2 + \tilde{\theta}_{\max})\|e(0)\| \] (26)

\[ |\dot{u}(4\tau_2)| \leq F_1(\ell, \gamma)\tilde{\theta}_{\max}^3 + \epsilon_{\tau_2} + F_2(\ell, \gamma)(\tilde{\theta}_{\max} + \epsilon_5)(\tilde{\theta}_{\max}^2 + \tilde{\theta}_{\max}) \] (27)

where

\[ F_1(\ell, \gamma) = O\left(\sqrt{\frac{\ell}{\gamma}}\right) + O\left(\frac{\ell}{\sqrt{\gamma}}\right) + O\left(\ell^2 \sqrt{\frac{\ell}{\gamma}}\right) \]

\[ F_2(\ell, \gamma) = O\left(\frac{\ell}{\sqrt{\gamma}}\right) + O\left(\ell \sqrt{\frac{\ell}{\gamma}}\right) \] (28)

The proof of Theorem 4 can be found in Appendix D.

The inequalities in (24)-(27) allow us to conclude the following compelling observations: For a given \( \|e(0)\|, \|\theta(0)\|, \tilde{\theta}_{\max} \) and \( \sigma \), we have that

\[ |\dot{u}(0)| \sim \frac{\gamma}{\ell} \] (29)

\[ |\dot{u}(0^+)| \sim \frac{\gamma}{\ell} \] (30)

\[ |\dot{u}(4\tau_1)| \sim \epsilon_{\tau_1}\frac{\gamma}{\ell} \] (31)

\[ |\dot{u}(4\tau_2)| \sim \frac{\ell^2 5}{\sqrt{\gamma}} \] (32)

The above relations lead to the following observations:
• If $\ell \geq \ell^*$ then the error dynamics will converge at a time constant $\tau_1$, an order of magnitude smaller than $\tau_2$, the time constant of reference model.

• If $\gamma$ is chosen such that the ratio of $\gamma$ is large, then the initial transients will be larger in $\dot{u}$, as observed in (29) and (30), followed by a smaller bound, nearly inversely proportional, on $\dot{u}$ at $t = 4\tau_2$ due to (32). Conversely, if $\gamma$ is chosen such that the ratio $\gamma$ is small, then the initial transients in $\dot{u}$ will be more benign, but could potentially grow large at later times.

That is, certain choices of $\gamma$ can lead to a water–bed effect, which is illustrated in Figure 1. Three different scenarios are compared, an open–loop reference model with $(\ell = 0, \gamma = 100)$ as a baseline for comparison, and two closed–loop reference models, one with the pair $(\ell = 10, \gamma = 1000)$ and the other with $(\ell = 10, \gamma = 10)$. While both closed–loop results have the same observer gain $\ell = 10$, they exhibit large transients in two different distinct regions as predicted by (29)-(32).

**Conjecture 1.** The above observations imply that given an observer gain $\ell$ and finite time $T$, there exist an optimal $\gamma_{optimal}$ such that

$$
\gamma_{optimal} = \arg \min_{\gamma} \int_0^T |\dot{u}(t)|^2 dt
$$

**D. Robustness to Time–Varying Uncertainties and Disturbances**

Consider the uncertain Linear Time Varying system

$$
\dot{x} = A_p(t)x(t) + bu + d(t) 
$$

where $d(t)$ is a bounded disturbance and $A_p(t)$ is now time varying with a bounded time–derivative. It is assumed that a time-varying vector $\theta^*(t)$ exists such that

$$
A_m = A_p(t) + b\theta^T(t)
$$
and $\theta_{\text{max}}$ exists such that $\|\theta^*(t)\| \leq \theta_{\text{max}}$.

**Lemma 5.** Given the uncertain system of equation in (33) with the reference model in (2), the controller in (4), the adaptive tuning law in (10)-(11) and (42), $\theta(0)$ chosen such that $\|\theta(0)\| \leq \theta + \varepsilon$, and with choices as in (13)-(14) the Lyapunov candidate in (9) converges exponentially to a set $\mathcal{E}$ given by

$$
\mathcal{E} \triangleq \{ (e, \tilde{\theta}) \mid \|e\|^2 \leq \beta_1 \tilde{\theta}_{\text{max}}^2 + \beta_2 \theta_d \tilde{\theta}_{\text{max}} + \beta_3 \|d\|^2, \|\tilde{\theta}\| \leq \tilde{\theta}_{\text{max}} \}
$$

(35)

where $\beta_2$ and $\beta_3$ are given in (76) with order of magnitudes given by

$$
\beta_2 = \mathcal{O}\left(\frac{1}{\gamma}\right) \quad \text{and} \quad \beta_3 = \mathcal{O}\left(\frac{1}{\ell^2}\right)
$$

(36)

with exponent $\alpha_3 \equiv \alpha_1/2$, where $\alpha_1$ is defined in (18).

**Proof.** see Appendix E.

**E. Simulation Study**

For this study a scalar time varying system in the presence of disturbances is to be controlled where

$$
\dot{x} = -a_p(t)x + u + d(t)
$$

(37)

with

$$
a_p(t) = \begin{cases} 1 & 0 \leq t < 20 \\ 1 + \frac{1}{4}(t - 20) & 20 \leq t < 24 \\ 2 & t \geq 24 \end{cases}
$$

$$
d(t) = \begin{cases} 0 & 0 \leq t < 20 \\ \text{disturbance, } \|d(t)\| \leq 0.1 & t \geq 20 \end{cases}
$$

(38)

The reference model to be followed is defined as

$$
\dot{x}_m = -1x_m + r + \ell(x - x_m)
$$

(39)

with $u = \theta(t)x + r$ where the update law comes from (5) and is defined as

$$
\dot{\theta} = -\gamma xep
$$

(40)

$$
p = \frac{1}{2(1 + \ell)}
$$

(41)

where $\ell$ and $\gamma$ are chosen as in Table 1.

<table>
<thead>
<tr>
<th>Eq. Number</th>
<th>Parameter</th>
<th>Open–Loop</th>
<th>Closed–Loop</th>
</tr>
</thead>
<tbody>
<tr>
<td>(39)</td>
<td>$\ell$</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>(40)</td>
<td>$\gamma$</td>
<td>100</td>
<td>1000</td>
</tr>
<tr>
<td>(41)</td>
<td>$p^*$</td>
<td>0.5</td>
<td>0.05</td>
</tr>
</tbody>
</table>

$p^*$ is a dependent variable from (41)

The simulations have 3 distinct regions of interest, the first 10 seconds where there are non-zero initial conditions in the state and the reference input is zero, the 10 second to 20 second range where a step command is given, and finally, the 20 to 35 second range where the uncertainty is now time varying and disturbances are added. Note the difference in the reference signals at the top of Figure 3. The black line is the open–loop reference model and the gray line is the closed–loop reference model. The closed–loop
Figure 3. (top) reference model trajectories $x_m$, (middle) state $x$, and (bottom) model following $e$.

Figure 4. (top) Control input $u$, (middle) rate of control input $\dot{u}$, and (bottom) adaptive parameter $\theta(t)$.

The reference model grows toward the initial condition of $x(0)$ and thus the model following error at the bottom of Figure 3 decreases more rapidly as compared to the classical control example. This has a low pass filtering affect on the adaptive parameter, as shown in bottom of Figure 4, as well as on the control input and rate of control input also in Figure 4. It may seem that the control gain of $\gamma = 1000$ is large, however, notice from equation (40) that it is the product $\gamma p$ that determines the rate at with $\dot{\theta}$ can change and for both the closed–loop and the open–loop examples the product just mentioned has a value of 10.
IV. Conclusions

Using the extra design freedom available in closed-loop reference models, we designed new adaptive identifiers, observers, and controllers that are (a) stable, and (b) have improved transient properties. The transients however can not be uniformly improved. While the Luenberger gain controls the rate of convergence of the model following error, the ratio of adaptive learning rate to Luenberger gain is an important design consideration, and when chosen improperly can lead to large rates in the control signal.

References

A. Projection Operator

The $\Gamma$–Projection Operator for two vectors $\theta, y \in \mathbb{R}^k$, a convex function $f(\theta) \in \mathbb{R}$ and with symmetric positive definite tuning gain $\Gamma \in \mathbb{R}^{k \times k}$ is defined as

$$
\text{Proj}_\Gamma(\theta, y, f) = \begin{cases} 
\Gamma y - \Gamma \nabla f(\theta)^T \Gamma f(\theta) & \text{if } f(\theta) > 0 \land y^T \Gamma \nabla f(\theta) > 0 \\
\Gamma y & \text{otherwise}
\end{cases}
$$

where $\nabla f(\theta) = \left( \frac{\partial f(\theta)}{\partial \theta_1} \cdots \frac{\partial f(\theta)}{\partial \theta_k} \right)^T$. The projection operator was first introduced in [26] with extensions in [2] and for a detailed analysis of $\Gamma$–projection see [27].

**Theorem 6.** Given $\dot{\theta} = \text{Proj}_\Gamma(\theta, y, f)$, $\theta^* \in D_0 = \{\theta \in \mathbb{R}^k | f(\theta) \leq 0\}$, $\theta(t=0) \in D_1 = \{\theta \in \mathbb{R}^k | f(\theta) \leq 1\}$ and $f(\theta) : \mathbb{R}^k \to \mathbb{R}$ is convex

$\theta(t) \in D_1 \forall t \geq 0$ and

$$(\theta - \theta^*)^T (\Gamma^{-1} \text{Proj}_\Gamma(\theta, y, f) - y) \leq 0.$$ (44)

**Proof.** See [27] □

B. Proof of Lemma 2

**Lemma 7** ([28, Lemma 1]). Any Hurwitz matrix $A_m \in \mathbb{R}^{n \times n}$ with constants $a$ and $\sigma$ as defined in (12) satisfies the following bound for the matrix exponential

$$\|\exp(A_m \tau)\| \leq m_{\epsilon} \exp((-\sigma + \epsilon a)\tau)$$

where $m_{\epsilon} = \frac{3}{2} \left(1 + \frac{2}{\epsilon}\right)^{n-1}$ and $\epsilon > 0$. The proof follows directly from [28].

**Corollary 8.** Setting $\epsilon = \sigma / (2a)$ the following holds

$$\|\exp(A_m \tau)\| \leq m \exp\left(-\frac{\sigma}{2} \tau\right),$$ (45)

where $m = \frac{3}{2} \left(1 + 4 \kappa\right)^{n-1}$ and $\kappa = \frac{a}{\sigma}$.

**Lemma 9.** For any diagonal matrix $L = -I_{n \times n}$ the following bound hold for the matrix exponential

$$\|\exp(L \tau)\| \leq \exp(-\ell \tau)$$ (46)

The proof follows from [29, Section 2].

**Proof of Lemma 2.** Beginning with the integral form of Lyapunov’s equation in (6)

$$P = \int_0^\infty \exp(A_m \tau) \exp(A_m \tau) \, d\tau.$$ 

Due to our choice of $L$, $A_m$ and $L$ commute, thus $\exp(A_m + L) = \exp(A_m) \exp(L)$ and

$$P = \int_0^\infty \exp(A_m \tau) \exp(L^T \tau) \exp(A_m \tau) \exp(L \tau) \, d\tau.$$ (47)

Using the bound in (45) and (46) the integral just above can be upper bounded as

$$\|P\| \leq \frac{m^2}{\sigma + 2\ell}.$$ (48) □
C. Proof of Theorem 3

Proof. Recall the Lyapunov candidate in (9), Taking its time derivative one has that

\[ \dot{V} = -\|e\|^2 \leq -\frac{1}{\|P\|} V + \frac{1}{\|P\|\gamma}\hat{\theta}_{\max}^2. \]

where Using the upper bound on \( P \) from (15)

\[ \dot{V} \leq -\alpha_1 V + \alpha_2 \]  \hspace{1cm} (49)

with \( \alpha_1 \) defined in (18) and \( \alpha_2 \triangleq \sigma + 2\ell \hat{\theta}_{\max}^2 \). Using the Gronwall Bellman Inequality, (49) implies that in

\[ \lim_{t \to \infty} e(t) P e(t) \leq \frac{1}{\gamma}\hat{\theta}_{\max}^2 \]

and using the bound in (15). \( \lim_{t \to \infty} \|e(t)\|^2 \leq \beta_1 \hat{\theta}_{\max}^2 \) where

\[ \beta_1 \triangleq \frac{2(s + \ell)}{\gamma}. \]  \hspace{1cm} (50)

D. Transient Analysis

Taking the time derivative of \( u \) in (4)

\[ \dot{u}(t) = -b P e(t) x^T(t)\gamma I_{n \times n} x(t) + \theta^T (A_p x(t) + b (\theta^T x(t) + r(t)) + \dot{r}(t). \]  \hspace{1cm} (51)

Substitution of the upper bound on \( P \) from (15) and defining \( g \triangleq \frac{\gamma}{\sigma + 2\ell} \) results in the following bound

\[ |\dot{u}(t)| \leq gm^2 \|b\| \|e(t)\| \|x(t)\|^3 \]

\[ + \|\theta(t)\| (\|A_p\| + \|b\|\|\theta(t)\|) \|x(t)\|. \]  \hspace{1cm} (52)

A. Bound at \( t = 0 \)

With direct substitution of the initial conditions for \( e(0) \) and \( \theta(0) \)

\[ |\dot{u}(0)| \leq O \left( \frac{\gamma}{\sigma + \ell} \|e(0)\|^3 \right) \]

\[ + O(1)(\|\theta(0)\|^2 + \|\theta(0)\|\|e(0)\|). \]  \hspace{1cm} (53)

B. Bound at \( t = 0^+ \)

Lemma 10. Given \( r \) and \( x_m \) satisfying Assumption 1

\[ \|x(t)\| \leq \|e(0)\| \exp(a_\theta t) \]  \hspace{1cm} (54)

where \( a_\theta \triangleq \|A_m + b\hat{\theta}_{\max}\| \geq 0. \)

Proof. The bound in (54) follows from a direct application of finite time stability theory [30, Theorem 8.14b]

Remark 1. Note that the exponent in (54) is non–negative and therefore not stable. However, given our choice of \( l \geq l^* \) in (22) the error dynamics converge much faster than the reference dynamics and therefore the finite time stability of the state vector can be exploited for small time scales.
Recall the dynamics of the reference model following error (8): $\dot{e} = A_m e + b \dot{\theta} x$. Using the integral transform of LTI systems and the bound for $\exp(A_m)$ from (45) and (46):

$$
\|e(t)\| \leq m e(0) \exp\left(-\frac{1}{\tau_1} t\right)
+ m b \|\tilde{\theta}_{\text{max}}\| \int_0^\infty \exp\left(-\frac{1}{\tau_1} (t - \tau)\right) \|x(\tau)\| d\tau
$$

(55)

where $\tau_1$ is defined in (19). After substitution of (54) and integrating

$$
\|e(t)\| \leq m e(0) \exp\left(-\frac{1}{\tau_1} t\right)
+ \epsilon_1 \left(\exp(\alpha_0 t) - \exp\left(-\frac{1}{\tau_1} t\right)\right)
$$

(56)

where $\epsilon_1$ is defined in (21). From (54) and (56) we can conclude that $\|x(0^+\| = \|e(0)\|$ and $\|e(0^+)\| = m \|e(0)\|$ and finally substituting these into (52)

$$
|\dot{u}(0^+)| \leq O\left(\frac{\gamma}{\sigma + \ell}\right) \|e(0)\|^3
+ O(1)(\|\tilde{\theta}_{\text{max}}\|^2 + \|\tilde{\theta}_{\text{max}}\|)\|e(0)\|.
$$

(57)

C. Bound at $t = 4\tau_1$

Direct substitution of $t = 4\tau_1$ into (54) and (56) results in the following bounds

$$
\|x(4\tau_1)\| = \|e(0)\| (1 + \epsilon_1)
$$

$$
\|e(4\tau_1)\| = m \|e(0)\| \epsilon_2 + \epsilon_1 (1 + \epsilon_1 - \epsilon_2)
$$

(58)

where $\epsilon_1 = \exp(-4)$ and $\epsilon_2$ is defined in (21). This leads to the following bound:

$$
\|e(4\tau_1)\| \|x(4\tau_1)\|^2 \leq m \|e(0)\|^3 \epsilon_{\tau_1}
$$

(59)

where

$$
\epsilon_{\tau_1} = 3 \epsilon_1 \epsilon_2 + (4 \epsilon_1 + 3 \epsilon_1^2) \epsilon_1 + \epsilon_1.
$$

Using the bound in (59) and (58) along with (52) we conclude that

$$
|\dot{u}(4\tau_1)| \leq O\left(\frac{\epsilon_{\tau_1} \gamma}{\sigma + \ell}\right) \|e(0)\|^3
+ O(1 + \epsilon_1)(\tilde{\theta}_{\text{max}}^2 + \tilde{\theta}_{\text{max}})\|e(0)\|.
$$

(60)

D. Bound at $t = 4\tau_2$

From (49) and (50) we know that

$$
\|e(t)\|^2 \leq k_0 \exp\left(-\frac{\sigma + 2\ell}{m^2} t\right) + k_1
$$

(61)

where $k_0 = k_0' - k_1$,

$$
k_0' = \frac{2m^2(s + \ell)}{\sigma + 2\ell} \|e(0)\|^2 + \frac{2(s + \ell)}{\gamma} \|\tilde{\theta}(0)\|^2
$$

$$
k_1 = \frac{2(s + \ell)}{\gamma} \tilde{\theta}_{\text{max}}^2
$$

(62)

Also, note that since $\kappa \geq 1$ and $\sigma, \ell > 0$ then $\frac{\ell}{\sigma + 2\ell} \leq \frac{\ell}{\sigma}$. Taking the square root of (61) and noting that $\sqrt{c_1 + c_2^2} \leq \sqrt{c_1} + \sqrt{c_2}$ for all $c_1, c_2 > 0$

$$
\|e(t)\| \leq \sqrt{k_0'} \exp\left(-\frac{1}{\tau_2}\right) + \sqrt{k_1}
$$

(63)
where \( \tau_1 \) is defined in (20). Using (63), at time \( t = 4\tau_2 \) we have the following bound on \( e \)

\[
\|e(4\tau_2)\| \leq \sqrt{k_0^2\varepsilon_3} + \sqrt{k_1}
\]  

(64)

where \( \varepsilon_3 = \exp \left(-\frac{4\tau_2^2}{\ell^2} \right) \), and is guaranteed to be small for \( t \geq \tau^* \) and (22).

Recall the reference model dynamics \( \dot{x}_m = A_m x_m - L e \) where by Assumption 1 \( x_m(0) = 0 \), then

\[
x_m(t) = -\int_0^t \exp (A_m(t - \tau)) Le(\tau) d\tau
\]

(65)

and using the bound on \( \exp(A_m) \) from (45) results in

\[
\|x_m(t)\| \leq \ell_m \int_0^t \exp \left(-\frac{\sigma}{2}(t - \tau)\right) \|e(\tau)\| d\tau
\]

(66)

using the bound for \( e \) from (63) results in the following bound for \( x_m \)

\[
\|x_m(t)\| \leq \nu_1 \left( 1 - \exp \left(-\frac{1}{\tau^2}t\right) \right)
+ \nu_2 \left( \exp \left(-\frac{1}{\tau^2}t\right) - \exp \left(-\frac{1}{\tau^2}t\right) \right)
\]

(67)

where \( \nu_1 = \frac{2\ell_m \sqrt{k_0}}{\sigma} \) and \( \nu_2 = \frac{\ell_m \sqrt{k_0}}{2\tau^2 - \sigma} \). Substitution of \( t = 4\tau_2 \) into (67) results in

\[
\|x_m(4\tau_2)\| \leq \nu_1 + \nu_2 \varepsilon_4
\]

(68)

where \( \varepsilon_4 = |\varepsilon_2 - \varepsilon_3| \) Using (64) and (68) one can conclude that

\[
\|e(4\tau_2) + x_m(4\tau_2)\| \leq \varepsilon_5 + \sqrt{k_1} + \nu_1
\]

(69)

where \( \varepsilon_5 = \sqrt{k_0} \varepsilon_3 + \nu_2 \varepsilon_4 \). The following bound also directly holds

\[
\|e(4\tau_2)\| \|e(4\tau_2) + x_m(4\tau_2)\|^2 \leq \varepsilon_{\tau_2} + \left(\sqrt{k_1} + \nu_1\right)^2 \sqrt{k_1}
\]

(70)

where

\[
\varepsilon_{\tau_2} = \left( \varepsilon_5^2 + 2\varepsilon_5 \left( \sqrt{k_1} + \nu_1 \right) \right) \left( \sqrt{k_0} \varepsilon_3 + \sqrt{k_1} \right)
+ \left( \sqrt{k_1} + \nu_1 \right)^2 \sqrt{k_0} \varepsilon_3.
\]

Using (69), (70) and (52)

\[
\|e(4\tau_2)\| \|e(4\tau_2) + x_m(4\tau_2)\|^2 \leq \left( O \left( \sqrt{\frac{\ell}{\gamma}} \right) + O \left( \ell \sqrt{\frac{\ell}{\gamma}} \right) + O \left( \ell^2 \sqrt{\frac{\ell}{\gamma}} \right) \right) \tilde{\theta}_\max^3 + \varepsilon_{\tau_2}
\]

with

\[
\|\dot{u}(4\tau_2)\| \leq F_1(\ell, \gamma) \tilde{\theta}_\max^3 + \varepsilon_{\tau_2}
+ F_2(\ell, \gamma) \tilde{\theta}_\max + \varepsilon_5 \tilde{\theta}_\max
\]

(71)

where

\[
F_1(\ell, \gamma) = O \left( \sqrt{\frac{\ell}{\gamma}} \right) + O \left( \ell \sqrt{\frac{\ell}{\gamma}} \right) + O \left( \ell^2 \sqrt{\frac{\ell}{\gamma}} \right)
\]

\[
F_2(\ell, \gamma) = O \left( \sqrt{\frac{\ell}{\gamma}} \right) + O \left( \ell \sqrt{\frac{\ell}{\gamma}} \right)
\]

(72)
E. Proof of Theorem 5

Proof. Taking the time derivative of the Lyapunov candidate in (9), substitution of the update law from (10) and the error dynamics in (8), the derivative of the lyapunov function can be upper bounded as

\[ \dot{V} \leq -\alpha_3 V + \alpha_4 \]  

(73)

where \( \alpha_3 \triangleq \alpha_1 / 2 \) was previously defined in (18) and

\[ \alpha_4 \triangleq \frac{\sigma + 2\ell \hat{\theta}_{\max}^2 + 2\gamma \hat{\theta}_{\max}^*}{\gamma} \left( \frac{m^2}{\sigma + 2\ell} \right)^2 \| \dot{d}(t) \|^2, \]  

(74)

Following the same procedure as in Appendix C we conclude that

\[ \lim_{t \to \infty} \| e(t) \|^2 \leq \beta_1 \hat{\theta}_{\max}^2 + \beta_2 \| \dot{\theta}^* \| \hat{\theta}_{\max} + \beta_3 \| d \|^2 \]  

(75)

where

\[ \beta_2 \triangleq \frac{8m^2}{\sigma \gamma} \quad \text{and} \quad \beta_3 \triangleq \frac{4sm^6}{\sigma (\sigma + \ell)^2}. \]  

(76) \[ \square \]